SOLUTIONS PROBLEM SET #3

1. (b) $\Rightarrow$ (a)

LET $m(x,y,z) = t(x, t(x,y,z), z)$. THEN THE IDENTITIES

$m(x,x,y) \approx m(x,y,x) \approx m(y,x,x) \approx x$

HOLD IN V, SO V IS CONGRUENCE-DISTRIBUTIVE. IT'S CONGRUENCE-PERMUTABLE BECAUSE

$t(x,x,y) \approx t(y,x,x) \approx y$ ARE IDENTITIES

(a) $\Rightarrow$ (b).

ASSUME V IS ARITHMETICAL. LET $F = F_V(3)$ AND $\bar{x} = x^V$, $\bar{y} = y^V$, $\bar{z} = z^V$ BE THE THREE FREE GENERATORS OF F.

LET $\alpha = \Theta(\bar{x}, \bar{y})$, $\beta = \Theta(\bar{y}, \bar{z})$, $\gamma = \Theta(\bar{x}, \bar{z})$.

$\gamma \leq \alpha \vee \beta$, SO $\gamma = \gamma \wedge (\alpha \vee \beta) = (\gamma \wedge \alpha) \vee (\gamma \wedge \beta)$
$= (\gamma \wedge \alpha) ; (\gamma \wedge \beta) = (\gamma \wedge \beta) ; (\gamma \wedge \alpha)$.

LET $t(x,y,z)$ BE A TERM SUCH THAT

$\bar{x} \, (\gamma \wedge \beta) \, t^F(\bar{x}, \bar{y}, \bar{z}) \, (\gamma \wedge \alpha) \, \bar{z}$. THUS

$\bar{x} \equiv t^F(\bar{x}, \bar{y}, \bar{z}) \, (\Theta(\bar{x}, \bar{z}))$

$\bar{x} \equiv t^F(\bar{x}, \bar{y}, \bar{z}) \, (\Theta(\bar{y}, \bar{z}))$

$\bar{z} \equiv t^F(\bar{x}, \bar{y}, \bar{z}) \, (\Theta(\bar{x}, \bar{y}))$. SO

$x \approx t(x,y,x)$, $x \approx t(x,y,y)$, AND

$z \approx t(x,x,z)$ ARE IDENTITIES OF V.

2. (a) LET $\underset{\sim}{A}$ BE A PRIMAL ALGEBRA.
DEFINE $f: A^3 \to A$ BY CONDITION

$$f(a,b,c) = \begin{cases} c & \text{IF } a=b \text{ OR } a=c \\ a & \text{OTHERWISE} \end{cases}$$

FOR ALL $a,b \in A$ WE HAVE
$f(a,a,b) = b$ AND $f(a,b,a) = a$.
IF $a=b$, THEN $f(a,b,b) = a$ BY FIRST ALTERNATIVE OF DEFINITION OF f, AND IF $a \neq b$, THEN $f(a,b,b) = a$ BY THE SECOND ALTERNATIVE. SO FOR ALL $a,b \in A$, $f(a,b,b) = a$.
LET $t(x,y,z) \in Te(x,y,z)$ SUCH THAT, FOR ALL $a,b,c \in A$

$$t^{\underset{\sim}{A}}(a,b,c) = f(a,b,c).$$

THEN $t(x,x,y) \approx y$, $t(x,y,x) \approx x$, AND $t(y,x,x) \approx y$ ARE IDENTITIES OF $\underset{\sim}{A}$ AND HENCE ALSO OF $HSP(\underset{\sim}{A})$.

(b) FOR EACH $a \in \mathbb{Z}_p = \{0,1,\ldots,p-1\}$ LET $\bar{a}$ BE THE CONSTANT 0 IF $a=0$ OR THE CONSTANT TERM $\underbrace{1+1+\cdots+1}_{a}$ IF $a \neq 0$. THEN $\bar{a}^{\mathbb{Z}_p} = a$ FOR EACH $a \in \mathbb{Z}_p$.
THUS EVERY POLYNOMIAL IN $\mathbb{Z}_p[x_1,\ldots,x_m]$ IN m VARIABLES

$$g(x_1,\ldots,x_m) = \sum_{(i_1,\ldots,i_m)\in\omega^m} \bar{a}_{i_1,\ldots,i_m} x_1^{i_1}\cdots x_m^{i_m}$$

IS A TERM IN $Te_{\Sigma}(x_1,\dots,x_m)$. SO IT SUFFICES TO SHOW THAT, FOR EVERY $m \in \omega$ AND EVERY $h : \mathbb{Z}_p^m \to \mathbb{Z}_p$, THERE IS A POLYNOMIAL $q(x_1,\dots,x_m)$ SUCH THAT $q^{\mathbb{Z}_p} = h$.

WE HAVE ALREAD SHOWN THIS FOR $m=0$ WITH THE CONSTANT POLYNOMIALS $\overline{0}, \overline{1}, \dots, \overline{p-1}$.

$m=1$: LET $h : \mathbb{Z}_p \to \mathbb{Z}_p$. IT IS KNOWN THAT THERE IS A POLYNOMIAL $q(x)$ OF DEGREE p SUCH THAT $q^{\mathbb{Z}_p}(a) = h(a)$ FOR EACH $a \in \mathbb{Z}_p$. (BY LAGRANGE'S INTERPOLATION FORMULAS SEE EXERCISE III.6.12 IN HUNGERFORD.)

FOR EACH $a \in \mathbb{Z}_p$ LET $u_a(x) \in \mathbb{Z}_p[x]$ SUCH THAT, $\forall b \in \mathbb{Z}_p$

$$u_a^{\mathbb{Z}_p}(b) = \begin{cases} 1 & \text{IF } b=a \\ 0 & \text{OTHERWISE} \end{cases}$$

WE NOW PROCEED WITH THE INDUCTION

SUPPOSE $m \geq 2$ AND $\forall h : \mathbb{Z}_p^m \to \mathbb{Z}_p$ $\exists$ A POLYNOMIAL $q(x_1,\dots,x_m)$ SUCH THAT $q^{\mathbb{Z}_p} = h$. NOW LET $h : \mathbb{Z}_p^{m+1} \to \mathbb{Z}_p$. FOR EACH $a \in \mathbb{Z}_p$ LET $h_a : \mathbb{Z}_p^m \to \mathbb{Z}_p$ SUCH THAT, $\forall b_1,\dots,b_m \in \mathbb{Z}_p$

$$h_a(b_1,\dots,b_m) = h(b_1,\dots,b_m,a).$$

LET $q_a(x_1,\ldots,x_m) \in \mathbb{Z}_p[x_1,\ldots,x_m]$ SUCH THAT $q_a^{\mathbb{Z}_p} = h_a$. FINALLY, TAKE

$$q(x_1,\ldots,x_{m+1}) = \sum_{a \in \mathbb{Z}_p} q_a(x_1,\ldots,x_m) \cdot u_a(x_{m+1})$$

FOR ALL $b_1,\ldots b_{m+1} \in \mathbb{Z}_p$ WE HAVE

$$\begin{aligned} q^{\mathbb{Z}_p}(b_1,\ldots,b_{m+1}) &= \sum_{a \in \mathbb{Z}_p} q_a^{\mathbb{Z}_p}(b_1,\ldots,b_m) \cdot^{\mathbb{Z}_p} u_a^{\mathbb{Z}_p}(b_{m+1}) \\ &= q_{b_{m+1}}^{\mathbb{Z}_p}(b_1,\ldots,b_m) \cdot^{\mathbb{Z}_p} 1 \\ &= h_{b_{m+1}}(b_1,\ldots,b_m) \\ &= h(b_1,\ldots,b_m,b_{m+1}). \end{aligned}$$

B. WE HAVE TO PROVE THE IMPLICATION FROM LEFT TO RIGHT. SUPPOSE $h : \underset{\frown}{A} \twoheadrightarrow \underset{\frown}{B}$ AND $\alpha \in \mathrm{Co}(\underset{\frown}{A})$. LET $\beta = \mathrm{KER}(h)$ AND ASSUME $a\,(\alpha;\beta;\alpha)\,a'$. THEN $\exists\, a_0, a_1 \in A$ SUCH THAT $a\,\alpha\,a_0\,\beta\,a_1\,\alpha\,a'$. THUS $h(a)\,h(\alpha)\,h(a_0)\,\beta\,h(a_1)\,h(\alpha)\,h(a')$. SINCE $\beta = \mathrm{KER}(h)$, $h(a_0) = h(a_1)$. SO $h(a)\,(h(\alpha);h(\alpha))\,h(a')$. BY ASSUMPTION $h(\alpha);h(\alpha) = h(\alpha)$. SO $h(a)\,h(\alpha)\,h(a')$. SO $\exists\, b, b' \in A$ SUCH THAT $b\,\alpha\,b'$ AND $h(a) = h(b)$ AND $h(a') = h(b')$. SO $a\,\beta\,b\,\alpha\,b'\,\beta\,a''$ HENCE $a\,(\beta;\alpha;\beta)\,a'$. SO $\alpha;\beta;\alpha \subseteq \beta;\alpha;\beta$.

4. (a) $\Rightarrow$ (b). LET $\alpha, \beta \in Co(\underset{\sim}{A})$ AND CONSIDER THE NATURAL MAP $\Delta_\beta : \underset{\sim}{A} \twoheadrightarrow \underset{\sim}{A}/\beta$. $\beta = \text{KER}(\Delta_\alpha)$. BY PROBLEM 3 $\alpha ; \beta ; \alpha \subseteq \beta ; \alpha ; \beta$. CONSIDERING THE NATURAL MAP $\Delta_\alpha : \underset{\sim}{A} \twoheadrightarrow \underset{\sim}{A}/\alpha$ GIVES THE REVERSE INCLUSION.

(b) $\Rightarrow$ (a). LET $h : \underset{\sim}{A} \twoheadrightarrow \underset{\sim}{B}$ AND $\alpha \in Co(\underset{\sim}{A})$. SINCE $\underset{\sim}{A}$ IS CONGRUENCE 3-PERMUTABLE, $\alpha ; \text{KER}(h) ; \alpha \subseteq \text{KER}(h) ; \alpha ; \text{KER}(h)$. SO $h(\alpha) \in Co(\underset{\sim}{B})$ BY PROBLEM 3.

5. BY THM 5.21

$(K \vee L)_\omega = HSP_u(K \cup L)_\omega \subseteq SPHS(K \cup L) = P_{SD}\, HS(K \cup L)$. LET $\underset{\sim}{A}$ BE A FINITE SUBDIRECTLY IRREDUCIBLE MEMBER OF $K \vee L$. THEN

$$\underset{\sim}{A} \in HS(K \cup L) = HS(K) \cup HS(L) = K \cup L \quad \boxtimes$$