**Math 589, Spring 2002** **Problem Set 3** **Due March 13 , 2002**

Corrected May 7, 2002

1. [Pixley] A class of $\Sigma$-algebras $\mathsf{K}$ is said to be *arithmetical* if it is both congruence-permutable and congruence-distributive. Prove that for any variety $\mathsf{V}$ the following are equivalent.
   (a) $\mathsf{V}$ is arithmetical.
   (b) There is a $t(x,y,z) \in \mathrm{Te}_\Sigma(x,y,z)$ such the following identities hold in $\mathsf{V}$.
$$t(x,y,x) \approx x, \qquad t(x,x,y) \approx t(y,x,x) \approx y.$$

2. A $\Sigma$-algebra $\boldsymbol{A}$ is *primal* if every operation on the universe $A$ of $\boldsymbol{A}$ is a term function, i.e., for every $n \in \omega$ and every $h\colon A^n \to A$, there is a $\Sigma$-term $t(x_0,\dots,x_{n-1})$ such that, for all $a_0,\dots,a_{n-1} \in A^n$, $h(a_0,\dots,a_{n-1}) = t^{\boldsymbol{A}}(a_0,\dots,a_{n-1})$.
   (a) Prove that the variety generated by a primal algebra is arithmetical.
   (b) Prove that, for any prime $p$, the prime field $\boldsymbol{\mathbb{Z}}_p = \langle \mathbb{Z}_p, +, \cdot, -, 0, 1\rangle$ is primal.

3. Assume $h\colon \boldsymbol{A} \twoheadrightarrow \boldsymbol{B}$ and $\alpha \in \mathrm{Co}(\boldsymbol{A})$. Prove that $h(\alpha) \in \mathrm{Co}(\boldsymbol{B})$ iff
$$\alpha \mathbin{;} \mathrm{rker}(h) \mathbin{;} \alpha \subseteq \mathrm{rker}(h) \mathbin{;} \alpha \mathbin{;} \mathrm{rker}(h).$$
   [*Note:* One of the two implications was proved in class. Prove the other one.]

4. Let $\boldsymbol{A}$ be a $\Sigma$-algebra. Prove that the following are equivalent.
   (a) For every $\Sigma$-algebra $\boldsymbol{B}$, for every $h\colon \boldsymbol{A} \twoheadrightarrow \boldsymbol{B}$ and for every $\alpha \in \mathrm{Co}(\boldsymbol{A})$, $h(\alpha) \in \mathrm{Co}(\boldsymbol{B})$.
   (b) $\boldsymbol{A}$ is congruence 3-permutable, i.e., for all $\alpha, \beta \in \mathrm{Co}(\boldsymbol{A})$, $\alpha \mathbin{;} \beta \mathbin{;} \alpha = \beta \mathbin{;} \alpha \mathbin{;} \beta$.

5. Assume $\mathsf{V}$ is a locally finite, congruence distributive variety, and that $\mathsf{K}$ and $\mathsf{L}$ are subvarieties of $\mathsf{V}$. Let $\mathsf{K} \vee \mathsf{L}$ be the join of $\mathsf{K}$ and $\mathsf{L}$ in the lattice of subvarieties of $\mathsf{V}$, i.e., $\mathsf{K} \vee \mathsf{L} = \mathbf{H}\,\mathbf{S}\,\mathbf{P}(\mathsf{K} \cup \mathsf{L})$. Prove that every finite, subdirectly irreducible member of $\mathsf{K} \vee \mathsf{L}$ is either in $\mathsf{K}$ or in $\mathsf{L}$.