SOLUTIONS PROBLEM SET #2

#1 PROOF OF LEMMA 1. LET $\mathcal{B} = (\Pi \underset{\sim}{A}_i)/\mathcal{I}(\mathcal{U})$

$\langle \mathcal{B}, \hat{a}/\mathcal{I}(\mathcal{U}) \rangle \models \varphi(\hat{x})$

IFF $\sigma_k^{\mathcal{B}}(\hat{a}/\mathcal{I}(\mathcal{U})) = s_k^{\mathcal{B}}(\hat{a}/\mathcal{I}(\mathcal{U}))$ SOME $k \leq m$

OR $u_\ell^{\mathcal{B}}(\hat{a}/\mathcal{I}(\mathcal{U})) = v_\ell^{\mathcal{B}}(\hat{a}/\mathcal{I}(\mathcal{U}))$ SOME $\ell \leq m$

IFF $\langle \sigma_k^{\Pi \underset{\sim}{A}_i}(\hat{a}), s_k^{\Pi \underset{\sim}{A}_i}(\hat{a}) \rangle \in \mathcal{I}(\mathcal{U})$ SOME $k \leq m$

OR $\langle u_\ell^{\Pi \underset{\sim}{A}_i}(\hat{a}), v_\ell^{\Pi \underset{\sim}{A}_i}(\hat{a}) \rangle \notin \mathcal{I}(\mathcal{U})$ SOME $\ell \leq m$

IFF (SINCE $\mathcal{U}$ IS ULTRAFILTER)

$\underline{EQ}(\sigma_k^{\Pi \underset{\sim}{A}_i}(\hat{a}), s_k^{\Pi \underset{\sim}{A}_i}(\hat{a})) \in \mathcal{U}$ SOME $k \leq m$

OR $\underline{EQ}(u_\ell^{\Pi \underset{\sim}{A}_i}(\hat{a}), v_\ell^{\Pi \underset{\sim}{A}_i}(\hat{a})) \notin \mathcal{U}$ SOME $\ell \leq m$

PROOF OF LEMMA 2. $\forall i \in I$

$\langle \underset{\sim}{A}_i, \hat{a}_i \rangle \models \varphi(\hat{x})$

IFF $\sigma_k^{\underset{\sim}{A}_i}(\hat{a}_i) = s_k^{\underset{\sim}{A}_i}(\hat{a}_i)$ SOME $k \leq m$

OR $u_\ell^{\underset{\sim}{A}_i}(\hat{a}_i) \neq v_\ell^{\underset{\sim}{A}_i}(\hat{a}_i)$ SOME $\ell \leq m$

IFF $i \in \underline{EQ}(\sigma_k^{\Pi \underset{\sim}{A}_i}(\hat{a}), s_k^{\Pi \underset{\sim}{A}_i}(\hat{a}))$ SOME $k \leq m$

OR $i \notin \underline{EQ}(u_\ell^{\Pi \underset{\sim}{A}_i}(\hat{a}), v_\ell^{\Pi \underset{\sim}{A}_i}(\hat{a}))$ SOME $\ell \leq m$

THUS $\{i \in I : \langle \underset{\frown}{A}_i, \hat{a}_i \rangle \models \varphi(\hat{x})\} =$

$$\bigcup_{k \le m} EQ\left(\tau_k^{\Pi \underset{\frown}{A}_i}(\hat{\vec{a}}), \nu_k^{\Pi \underset{\frown}{A}_i}(\hat{\vec{a}})\right) \cup$$

$$\cup \bigcup_{l \le m} \overline{EQ\left(\eta_l^{\Pi \underset{\frown}{A}_i}(\hat{\vec{a}}), \nu_l^{\Pi \underset{\frown}{A}_i}(\hat{\vec{a}})\right)}$$

SINCE $\mathcal{U}$ IS AN ULTRAFILTER

$X_1 \cup \cdots \cup X_p \in \mathcal{U}$ IFF $X_k \in \mathcal{U}$ SOME $k \le p$.

SO BY THE LEMMAS, $\forall\ \hat{a} \in (\Pi A_i)^k$

(*) $\langle \underset{\frown}{B}, \hat{\vec{a}}/\Phi(\mathcal{U}) \rangle \models \varphi(\hat{x})$ IFF

$$\{i \in I : \langle \underset{\frown}{A}_i, \hat{a}_i \rangle \models \varphi(\hat{x})\} \in \mathcal{U}$$

ASSUME $\{i \in I : \underset{\frown}{A}_i \models \varphi(\vec{x})\} \in \mathcal{U}$.

$$\{i \in I : \underset{\frown}{A}_i \models \varphi(\vec{x})\} = \bigcap_{\hat{a} \in (\Pi A_i)^k} \{i \in I : \langle \underset{\frown}{A}_i, \hat{\vec{a}} \rangle \models \varphi(\vec{x})\}$$

SO $\forall\ \hat{\vec{a}} \in (\Pi A_i)^k \left(\{i \in I : \langle \underset{\frown}{A}_i, \hat{\vec{a}} \rangle \models \varphi(\vec{x})\} \in \mathcal{U}\right)$

THUS BY (*)

$$\forall \hat{\vec{a}} \in (\Pi A_i)^k \left(\langle \underset{\frown}{B}, \hat{\vec{a}} \rangle / \Phi(\mathcal{U})\right) \models \varphi(\vec{x})$$

IE $(\Pi \underset{\frown}{A}_i)/\Phi(\mathcal{U}) \models \varphi(\hat{x})$.

NOW ASSUME $\{i \in I : \underset{\frown}{A}_i \models \varphi\} \notin \mathcal{U}$.

LET $J = \{ i \in I : \mathfrak{A}_i \not\models \varphi(\hat{x}) \}$. THEN $J \in \mathcal{U}$ SINCE $\bar{J} \notin \mathcal{U}$. FOR EACH $j \in J$ LET $\hat{a}_j = \langle a_{j0}, \dots, a_{j(n-1)} \rangle \in A_j^n$ SUCH THAT $\langle \mathfrak{A}_j, \hat{a}_j \rangle \not\models \varphi(\hat{x})$. LET $\hat{\hat{a}} \in (\prod_{i \in I} A_i)^n$ SUCH THAT, FOR EACH $j \in J$, THE jTH ROW OF $\hat{\hat{a}}$ IS THE $\hat{a}_j$ ABOVE; FOR $i \notin J$ THE ROW CAN BE ANYTHING. THEN $\{ i \in I : \langle \mathfrak{A}_i, \hat{a}_i \rangle \models \varphi(\hat{x}) \} \subseteq \bar{J}$. SO THIS SET IS NOT IN $\mathcal{U}$. THUS BY (*) $\langle \mathfrak{B}, \hat{\hat{a}}/\Phi(\mathcal{U}) \rangle \not\models \varphi(\hat{x})$, AND HENCE

$$\prod \mathfrak{A}_i / \Phi(\mathcal{U}) \not\models \varphi(\hat{x}).$$

#2 WE FIRST GIVE A DIRECT PROOF. ASSUME $E \cup \Gamma \vdash \varepsilon$ AND LET $\delta_1, \dots, \delta_m$ WITH $\delta_m = \varepsilon$ BE A PROOF OF ε FROM $E \cup \Gamma$. WE PROVE THAT $E \vdash \varepsilon$ BY INDUCTION ON THE NUMBER OF OCCURRENCES OF SUBSTITUTION INSTANCES OF EQUATIONS OF Γ IN THE PROOF. IF THERE ARE NONE, THEN IT IS ALREADY A PROOF OF ε FROM E ALONE.

SUPPOSE THERE IS AT LEAST ONE SUCH OCCURRENCE. LET δ_i BE THE FIRST SUCH. THEN $\delta_i = \gamma(\hat{y})$ WITH $\gamma(\hat{x}) \in \Gamma$ AND $\hat{y} \in \mathrm{Te}_{\Sigma}(X)^m$ (ASSUME $\hat{x} = \langle x_0, \dots, x_{m-1} \rangle$).

LET $K_1(\hat{x}), \ldots, K_\ell(\hat{x})$ WITH $K_\ell(\hat{x}) = \gamma(\hat{x})$ BE A PROOF OF $\gamma(\hat{x})$ FROM E. (WE ASSUME $\hat{x}$ INCLUDES ALL VARIABLES OCCURRING IN THE PROOF). NOTE THAT BY DEFINITION OF PROOF, $K_1(\hat{q}), \ldots, K_\ell(\hat{q})$ IS A PROOF OF $\gamma(\hat{q})$ FROM E. THEN

$$\delta_1, \ldots, \delta_{i-1}, K_1(\hat{q}), \ldots, \underbrace{K_\ell(\hat{q})}_{\delta_i}, \delta_{i+1}, \ldots, \delta_m \text{ IS}$$

A PROOF OF ε FROM $E \cup \Pi$ THAT CONTAINS ONE LESS OCCURRENCE OF A SUBSITUTION INSTANCE OF AN EQUATION IN Π. SO BY IND. HYP. $E \vdash \varepsilon$.

WE NOW GIVE AN INDIRECT PROOF USING THE SOUNDNESS AND COMPLETENESS THEOREMS.

$Cn(E) = \{\varepsilon : E \models \varepsilon\}$. BY COMPLETENESS AND SOUNDNESS THEOREMS $Cn(E) = \{\varepsilon : E \vdash \varepsilon\}$. BY THM 4.7 Cn IS A CLOSURE OPERATOR ON $Te^2_Z(X)$. THUS

(1) $E \subseteq Cn(E)$
(2) $Cn(Cn(E)) = Cn(E)$
(3) $E \subseteq F \Rightarrow Cn(E) \subseteq Cn(F)$.

<u>CLAIM</u>. $\Pi \subseteq Cn(E) \Rightarrow Cn(E \cup \Pi) = Cn(E)$.

<u>PROOF</u>. $Cn(E) \subseteq Cn(E \cup \Pi)$ BY (3). FROM $\Pi \subseteq Cn(E)$ AND $E \subseteq Cn(E)$ WE GET $E \cup \Pi \subseteq Cn(E)$. SO BY (3) AND (2) $Cn(E \cup \Pi) \subseteq Cn\,Cn(E) = Cn(E)$ $\boxtimes$ CLAIM.

SUPPOSE $E \vdash \gamma$ FOR EACH $\gamma \in \Gamma$ AND $E \cup \Gamma \vdash \varepsilon$. THEN $\Gamma \subseteq Cn(E)$ AND $\varepsilon \in Cn(E \cup \Gamma)$. BUT $Cn(E \cup \Gamma) = Cn(E)$ BY CLAIM, SO $\varepsilon \in Cn(E)$ AND HENCE $E \vdash \varepsilon$.

#3. WE FIRST PROVE THE LEMMA

(*) $E \cup \{x \cdot x \approx e\} \vdash x^{-1} \approx x$.

HERE IS THE "MATH 504 STYLE" PROOF

$$x^{-1} = x^{-1} \cdot e = x^{-1} \cdot (x \cdot x) = (x^{-1} \cdot x) \cdot x = e \cdot x = x.$$

WE CONVERT THIS TO A FORMAL EQUATIONAL PROOF.

1.	$x^{-1} \cdot e \approx x^{-1}$	(AXIOM) (SUBSTITUTE x^{-1} FOR x IN $x \cdot e \approx x$)
2.	$x^{-1} \approx x^{-1} \cdot e$	1, (SYM)
3.	$x \cdot x \approx e$	(AXIOM)
4.	$e \approx x \cdot x$	3, (SYM)
5.	$x^{-1} \approx x^{-1}$	(TAUT)
6.	$x^{-1} \cdot e \approx x^{-1} \cdot (x \cdot x)$	4, 5, (REPL)
7.	$x^{-1} \approx x^{-1} \cdot (x \cdot x)$	2, 6, (TRAN)
8.	$x^{-1} \cdot (x \cdot x) \approx (x^{-1} \cdot x) \cdot x$	(AXIOM)
9.	$x^{-1} \approx (x^{-1} \cdot x) \cdot x$	7, 8 (TRAN)
10.	$x^{-1} \cdot x \approx e$	(AXIOM)
11.	$x \approx x$	(TAUT)
12	$(x^{-1} \cdot x) \cdot x \approx e \cdot x$	10, 11 (REPL)
13	$x^{-1} \approx e \cdot x$	9, 12, (TRAN)
14.	$e \cdot x \approx x$	(AXIOM)
15	$x^{-1} \approx x$	13, 14 (TRAN)

WE NEXT PROVE

(**) $E \cup \{x^{-1} \approx x,\ (x\cdot y)^{-1} \approx y^{-1}\cdot x^{-1}\} \vdash x\cdot y \approx y\cdot x.$

FIRST THE "MATH 504 STYLE" PROOF

$$x\cdot y = (x\cdot y)^{-1} = y^{-1}\cdot x^{-1} = y\cdot x.$$

1.	$x\cdot y \approx (x\cdot y)^{-1}$	(AXIOM) (SUBSTITUTE $x\cdot y$ FOR x IN $x^{-1} \approx x$)
2.	$(x\cdot y)^{-1} \approx x\cdot y$	1, (TRAN)
3.	$x\cdot y \approx y^{-1}\cdot x^{-1}$	1, 2, (TRAN)
4.	$y^{-1} \approx y$	(AXIOM) (SUBSTITUTE y FOR x IN $x^{-1} \approx x$)
5.	$x^{-1} \approx x^{-1}$	(TAUT)
6.	$y^{-1}\cdot x^{-1} \approx y\cdot x^{-1}$	4, 5, (REPL)
7.	$x^{-1} \approx x$	(AXIOM)
8.	$y \approx y$	(TAUT)
9.	$y\cdot x^{-1} \approx y\cdot x$	7, 8 (REPL)
10.	$y^{-1}\cdot x^{-1} \approx y\cdot x$	6, 9 (TRAN)
11	$x\cdot y \approx y\cdot x$	3, 10, (TRAN)

BY ASSUMPTION

(**) $\qquad E \vdash (x\cdot y)^{-1} \approx y^{-1}\cdot x^{-1}$

LET $\Gamma = \{x^{-1} \approx x,\ (x\cdot y)^{-1} \approx y^{-1}\cdot x^{-1}\}$

$E \cup \{x\cdot x \approx e\} \vdash \gamma$ FOR EACH $\gamma \in \Gamma$ BY (*) AND (**). $E \cup \{x\cdot x \approx e\} \cup \Gamma \vdash x\cdot y \approx y\cdot x$ BY (**)

SO $E \cup \{x\cdot x \approx e\} \vdash x\cdot y \approx y\cdot x$ BY PROB # 2.

#4 (a) ASSUME E IS INCONSISTENT. THEN EVERY MODEL OF E IS TRIVIAL, I.E., CONTAINS ONLY ONE ELEMENT. LET $\underset{\sim}{A} \in Mod(E)$ AND LET e BE ITS UNIQUE ELEMENT. LET $t(x,y) = x$ AND $s(x,y) = y$. THEN FOR ALL $a, b \in A$ $t^{\underset{\sim}{A}}(a,b) = e = s^{\underset{\sim}{A}}(a,b)$. SO $\underset{\sim}{A}$ IS A MODEL OF $x \approx y$. THUS $E \models x \approx y$ AND HENCE $E \vdash x \approx y$ BY COMPLETENESS THEOREM. SUPPOSE NOW THAT $E \vdash x \approx y$. THEN $E \models x \approx y$ BY SOUNDNESS THEOREM. SO EVERY MODEL OF E IS A MODEL OF $x \approx y$. BUT CLEARLY EVERY MODEL OF $x \approx y$ IS TRIVIAL.

(b) THE COMPACTNESS THEOREM SAYS THAT IF E IS INCONSISTENT, THEN SOME FINITE SUBSET E' OF E IS INCONSISTENT. SUPPOSE E IS INCONSISTENT, I.E. EVERY MODEL IS TRIVIAL. BY PART (a) $E \vdash x \approx y$. LET $\delta_1, \delta_2, \ldots, \delta_m$ BE A PROOF OF $x \approx y$ FROM E. LET $\delta_{i_1}, \ldots, \delta_{i_k}$ BE THE SUBSEQUENCE OF $\delta_1, \ldots, \delta_m$ SUCH THAT EACH δ_{i_j} IS A SUBSTITUTION INSTANCE OF AN EQUATION IN E; LET ε_j BE ONE OF THEM (THERE MAY BE MANY) AND LET $E' = \{\varepsilon_1, \ldots, \varepsilon_k\}$

THEN $\delta_1, \ldots, \delta_m$ IS A PROOF OF $X \approx Y$ FROM $\underline{E}'$. THUS $\underline{E}' \vdash X \approx Y$. SO $\underline{E}'$ IS INCONSISTENT

#5. BY THM 1.8 IT SUFFICES TO PROVE THAT H IS A STRICTLY ORDER REVERSING BIJECTION. FOR EACH $C \in \mathcal{C}_A$, $H(G(C)) = (H \circ G)(C) = C$, AND FOR EACH $D \in \mathcal{C}_B$, $G(H(D)) = (G \circ H)(D) = D$.
SO H IS A BIJECTION BETWEEN $\mathcal{C}_A$ AND $\mathcal{C}_B$ WITH INVERSE G. SINCE H, G IS A GALOIS CONNECTION, $C \subseteq C' \Rightarrow H(C) \supseteq H(C')$
CONVERSELY $H(C) \supseteq H(C') \Rightarrow$
$C = G(H(C)) \subseteq G(H(C'))\ C'$. SO H IS STRICTLY ORDER REVERSING.