

1. Let  $\langle \mathbf{A}_i : i \in I \rangle$  be a system of  $\Sigma$ -algebras, and let  $\mathcal{U}$  be an ultrafilter on  $I$ . Let  $\varphi$  be a formula of the form  $\varepsilon_1 \text{ or } \cdots \text{ or } \varepsilon_n \text{ or } (\text{not } \delta_1) \text{ or } \cdots \text{ or } (\text{not } \delta_m)$ , where  $\varepsilon_1, \dots, \varepsilon_n, \delta_1, \dots, \delta_m$  are  $\Sigma$ -equations. Prove that

$$\left( \prod_{i \in I} \mathbf{A}_i \right) / \Phi(\mathcal{U}) \models \varphi \quad \text{iff} \quad \{i \in I : \mathbf{A}_i \models \varphi\} \in \mathcal{U}.$$

[*Hint:* The solution cannot be easily obtained simply by generalizing the proof of Lemma 3.38. First of all the class of models of an equational clause is not necessarily closed under homomorphic images or direct products (integral domains, for example). A somewhat different argument is required. Here is how I suggest you proceed. A big part of the problem is to choose the right kind of simplifying notation.

Let  $I$  be an index set, which is normally infinite. We use vector notation to represent a “ $I$ -dimensional” vector over an arbitrary  $I$ -indexed system of sets  $\langle \mathbf{A}_i : i \in I \rangle$ . Thus  $\vec{a} = \langle a_i : i \in I \rangle \in \prod_{i \in I} \mathbf{A}_i$ . On the other hand, we will use “hats” to denote arbitrary finite sequences of elements of  $A$ , for example,  $\hat{a} = a_0, \dots, a_{k-1} \in A^k$ . Thus  $\hat{\vec{a}} = \vec{a}_0, \dots, \vec{a}_{k-1}$  is a finite sequence of  $I$ -vectors, i.e., a “ $I$  by  $k$ ” matrix of elements of the  $\mathbf{A}_i$ . For each  $j < k$ ,  $\vec{a}_j$  is the  $j$ -th column of the matrix, and for each  $i \in I$ ,  $\hat{a}_i = a_{i0}, \dots, a_{i(k-1)}$  is the  $i$ -th row. Let  $\varphi(\hat{x})$  be the equational clause

$$(t_1(\hat{x}) \approx s_1(\hat{x})) \text{ or } \cdots \text{ or } (t_n(\hat{x}) \approx s_n(\hat{x})) \text{ or } (\text{not}(u_1(\hat{x}) \approx v_1(\hat{x}))) \text{ or } \cdots \text{ or } (\text{not}(u_m(\hat{x}) \approx v_m(\hat{x}))),$$

where  $\hat{x} = x_0, \dots, x_{k-1}$  is a list of all the variables that occur in  $\varphi$ .

The intermediate notion of *satisfaction* is useful here. Let  $\mathbf{A}$  be a  $\Sigma$ -algebra and  $\hat{a} = a_0, \dots, a_{k-1} \in A^k$ . We say that  $\hat{a}$  *satisfies*  $\varphi(\hat{x})$  in  $\mathbf{A}$ , in symbols  $\langle \mathbf{A}, \hat{a} \rangle \models \varphi(\hat{x})$ , if

$$t_1^{\mathbf{A}}(\hat{a}) = s_1^{\mathbf{A}}(\hat{a}) \quad \text{or } \cdots \text{ or } \quad t_n^{\mathbf{A}}(\hat{a}) = s_n^{\mathbf{A}}(\hat{a}) \quad \text{or} \quad u_1^{\mathbf{A}}(\hat{a}) \neq v_1^{\mathbf{A}}(\hat{a}) \quad \text{or } \cdots \text{ or} \quad u_m^{\mathbf{A}}(\hat{a}) \neq v_m^{\mathbf{A}}(\hat{a}).$$

Thus  $\varphi(\hat{x})$  is universally valid in  $\mathbf{A}$  if, for every  $\hat{a} \in A^k$ ,  $\hat{a}$  satisfies  $\varphi(\hat{x})$  in  $\mathbf{A}$ .

Let  $\prod_{i \in I} \mathbf{A}_i$  be a system of  $\Sigma$ -algebras and  $\hat{\vec{a}} = \vec{a}_0, \dots, \vec{a}_{k-1} \in (\prod_{i \in I} \mathbf{A}_i)^k$ . Let  $\mathcal{U}$  be an ultrafilter on  $I$ , and let  $\hat{\vec{a}} / \Phi(\mathcal{U}) = \vec{a}_0 / \Phi(\mathcal{U}), \dots, \vec{a}_{k-1} / \Phi(\mathcal{U})$ . Prove the following two lemmas

**Lemma 1.**  $\langle (\prod_{i \in I} \mathbf{A}_i) / \Phi(\mathcal{U}), \hat{\vec{a}} / \Phi(\mathcal{U}) \rangle \models \varphi(\hat{x})$  iff

$$\begin{aligned} & \text{EQ}(t_1^{\prod \mathbf{A}_i}(\hat{\vec{a}}), s_1^{\prod \mathbf{A}_i}(\hat{\vec{a}})) \in \mathcal{U} \quad \text{or } \cdots \text{ or } \quad \text{EQ}(t_n^{\prod \mathbf{A}_i}(\hat{\vec{a}}), s_n^{\prod \mathbf{A}_i}(\hat{\vec{a}})) \in \mathcal{U} \\ & \quad \text{or} \quad \text{EQ}(u_1^{\prod \mathbf{A}_i}(\hat{\vec{a}}), v_1^{\prod \mathbf{A}_i}(\hat{\vec{a}})) \notin \mathcal{U} \quad \text{or } \cdots \text{ or} \quad \text{EQ}(u_m^{\prod \mathbf{A}_i}(\hat{\vec{a}}), v_m^{\prod \mathbf{A}_i}(\hat{\vec{a}})) \notin \mathcal{U}. \end{aligned}$$

**Lemma 2.**  $\{i \in I : \langle \mathbf{A}_i, \hat{a}_i \rangle \models \varphi(\hat{x})\} =$

$$\begin{aligned} & \text{EQ}(t_1^{\prod \mathbf{A}_i}(\hat{\vec{a}}), s_1^{\prod \mathbf{A}_i}(\hat{\vec{a}})) \cup \cdots \cup \text{EQ}(t_n^{\prod \mathbf{A}_i}(\hat{\vec{a}}), s_n^{\prod \mathbf{A}_i}(\hat{\vec{a}})) \\ & \quad \cup \overline{\text{EQ}(u_1^{\prod \mathbf{A}_i}(\hat{\vec{a}}), v_1^{\prod \mathbf{A}_i}(\hat{\vec{a}}))} \cup \cdots \cup \overline{\text{EQ}(u_m^{\prod \mathbf{A}_i}(\hat{\vec{a}}), v_m^{\prod \mathbf{A}_i}(\hat{\vec{a}}))}. \end{aligned}$$

(over)

Now for the solution of Problem 1. For the proof of the implication from right to left, use that fact that, for all  $\hat{a} \in (\prod_{i \in I} A_i)$ ,  $\{i \in I : \mathbf{A}_i \models \varphi(\hat{x})\} \subseteq \{i \in I : \langle \mathbf{A}_i, \hat{a}_i \rangle \models \varphi(\hat{x})\}$ . For the implication in the opposite direction, prove the contrapositive.

Assume  $\{i \in I : \mathbf{A}_i \models \varphi(\hat{x})\} \notin \mathcal{U}$ , and show that there is a  $\hat{a} \in (\prod_{i \in I} A_i)$  such that  $(\prod_{i \in I} \mathbf{A}_i) / \Phi(\mathcal{U}) \not\models \varphi(\hat{x})$ .

2. Let  $E$  and  $\Gamma$  be sets of  $\Sigma$ -equations such that  $E \vdash \gamma$  for every  $\gamma \in \Gamma$ . Prove that, for every  $\Sigma$ -equation  $\varepsilon$ , if  $E \cup \Gamma \vdash \varepsilon$ , then  $E \vdash \varepsilon$ .

[*Note:* The equations of  $\Gamma$  can be viewed as “lemmas” that are used in the “proof” of the “theorem”  $\varepsilon$  from the “hypotheses”  $E$ . To prove this directly one has to show that, given any proof  $\delta_1, \dots, \delta_m$  of  $\varepsilon$  from  $E \cup \Gamma$  (in the precise sense of Definition 4.2), one can replace each occurrence of a substitution instance  $\gamma'$  of an equation  $\gamma$  of  $\Gamma$  by a proof of  $\gamma'$  from  $E$ , thus obtaining a, generally much longer, proof of  $\varepsilon$  from  $E$  alone. A shorter, indirect proof can be obtained using the soundness and completeness theorems of equational logic, and the fact the logical consequence operation  $\text{Cn}$  is a closure operator (Theorem 4.7).]

3. Let  $E$  be the axioms of groups (of Type II). Prove that  $E \cup \{x \cdot x \approx e\} \vdash x \cdot y \approx y \cdot x$ .

[*Hint:* You have to prove that a proof of  $x \cdot y \approx y \cdot x$  (in the sense of Definition 4.2) exists. One way to do this is to just write it down like we did in class (or at least started to do) for  $(x \cdot y)^{-1} \approx y^{-1} \cdot x^{-1}$ . But this will be very long. It is better to first prove some lemmas and then use Problem #2. You might even want to use lemmas in the proof of a lemma. But for the lowest level lemmas you have to write out formal proofs in the sense of Definition 4.2. I suggest that you give an informal proof of  $x \cdot y \approx y \cdot x$  from  $E \cup \{x \cdot x \approx e\}$  as you would do in a beginning algebra course and then convert it to a formal proof. You can assume that  $E \vdash (x \cdot y)^{-1} \approx y^{-1} \cdot x^{-1}$ , and hence use  $(x \cdot y)^{-1} \approx y^{-1} \cdot x^{-1}$  as a lemma.]

4. Recall that a set  $E$  of  $\Sigma$ -equations is *inconsistent* if it has only trivial models.

- (a) Prove that  $E$  is inconsistent iff  $E \vdash x \approx y$ , where  $x$  and  $y$  are distinct variables.  
 (b) Use part (a) to obtain another proof of the compactness theorem for equational logic that does not use reduced products.

[*Hint:* For part (b) look at the proof that the closure relation  $\text{Cn}$  is finitary.]

5. Let  $A$  and  $B$  be sets and  $R \subseteq A \times B$ . Let  $H: \mathcal{P}(A) \rightarrow \mathcal{P}(B)$  and  $G: \mathcal{P}(B) \rightarrow \mathcal{P}(A)$  be the Galois connection defined by  $R$ . Let  $\mathcal{C}_A = \{C \subseteq A : (G \circ H)(C) = C\}$ , the closed subsets of  $A$  under  $G \circ H$ . Let  $\mathcal{C}_B = \{C \subseteq B : (H \circ G)(C) = C\}$ , the closed subsets of  $B$  under  $H \circ G$ . Prove that the complete lattices  $\langle \mathcal{C}_A, \subseteq \rangle$  and  $\langle \mathcal{C}_B, \subseteq \rangle$  are dually isomorphic under  $H$ . Specifically, prove that  $H$  is a bijection between  $\mathcal{C}_A$  and  $\mathcal{C}_B$  such that, for all  $C, C' \in \mathcal{C}_A$ ,  $C \subseteq C'$  iff  $H(C) \supseteq H(C')$ .