SOLUTIONS PROBLEM SET #1.

1. ASSUME $\mathcal{V}$ IS NONTRIVIAL. LET $\underset{\sim}{F} = \coprod_{\xi<\kappa} \underset{\sim}{F}_\xi$.
LET $\amalg_\xi : \underset{\sim}{F}_\xi \to \underset{\sim}{F}$ BE THE COPROJECTION FOR EACH $\xi<\kappa$. BY DEFINITION OF COPRODUCT, $\underset{\sim}{F} \in \mathcal{V}$.

FOR EACH $\xi<\kappa$, LET $\overline{x}_\xi = \amalg_\xi(x_0^{\mathcal{V}})$.

<u>CLAIM</u>. $\overline{x}_\zeta \neq \overline{x}_\eta$ FOR ALL $\zeta, \eta<\kappa$ SUCH THAT $\zeta\neq\eta$.

<u>PROOF</u> OF CLAIM. LET $\underset{\sim}{A}$ BE A NONTRIVIAL ALGEBRA OF $\mathcal{V}$, AND LET $a, b \in A$ WITH $a \neq b$. FOR EACH $\xi<\kappa$, LET $h_\xi : \underset{\sim}{F}_\xi \to \underset{\sim}{A}$ BE THE (UNIQUE) HOMOMORPHISM SUCH THAT $h_\xi(x_0^{\mathcal{V}}) = \begin{cases} a & \text{IF } \xi=\zeta \\ b & \text{IF } \xi\neq\zeta \end{cases}$. BY THE COPRODUCT PROPERTY $\exists\, h : \underset{\sim}{F} \to \underset{\sim}{A}$ SUCH THAT $h_\xi = h \circ \amalg_\xi \;\forall\, \xi<\kappa$. IN PARTICULAR, $h(\overline{x}_\zeta) = h(\amalg_\zeta(x_0^{\mathcal{V}})) = h_\zeta(x_0^{\mathcal{V}}) = a \neq b = h(\overline{x}_\eta)$. SO $\overline{x}_\zeta \neq \overline{x}_\eta$ $\boxtimes$ CLAIM.

LET $\overline{X}_\kappa = \{\overline{x}_\xi : \xi<\kappa\}$. BY THE CLAIM $|\overline{X}_\kappa| = \kappa$. AND $Sg^{\underset{\sim}{F}}(\overline{X}_\kappa) = Sg^{\underset{\sim}{F}}\left(\bigcup_{\xi<\kappa} Sg^{\underset{\sim}{F}}(\overline{x}_\xi)\right) = Sg^{\underset{\sim}{F}}\left(\bigcup_{\xi<\kappa} \amalg_\xi(F_\xi)\right) = F$, AGAIN BY DEFINITION OF THE COPRODUCT. SO $\overline{X}_\kappa$ GENERATES $\underset{\sim}{F}$. TO SHOW $\underset{\sim}{F} \cong \underset{\sim}{F}_{\mathcal{V}\kappa}(\mathcal{V})$ IT THUS SUFFICES TO SHOW THAT $\underset{\sim}{F}$ HAS THE UNIVERSAL MAPPING PROPERTY OVER $\mathcal{V}$ WRT $\overline{X}_\kappa$.

LET $\underset{\sim}{A} \in V$ AND LET $h : \overline{X}_\kappa \to A$.
$\exists \xi < \kappa$ $\exists h_\xi : \underset{\sim}{F}_\xi \to \underset{\sim}{A}$ SUCH THAT
$h_\xi(x_0^V) = h(\overline{x}_\xi)$. SO BY THE COPRODUCT
PROPERTY $\exists h^* : \underset{\sim}{F} \to \underset{\sim}{A}$ SUCH THAT
$h_\xi = h^* \circ \amalg_\xi$ FOR EACH $\xi < \kappa$. THEN
$h^*(\overline{x}_\xi) = h^*(\amalg_\xi(x_0^V)) = h_\xi(x_0^V) = h(\overline{x}_\xi)$
FOR EACH $\xi < \kappa$. SO $\underset{\sim}{F}$ HAS THE
UNIVERSAL MAPPING PROPERTY. HENCE
$\coprod_{\xi<\kappa} \underset{\sim}{F}_\xi \cong \underset{\sim}{F}_{V_\kappa}(V)$.

2. LET $\mathcal{F}$ BE THE FILTER GENERATED BY $\mathcal{K}$. BY LEM. 3.31, FOR EACH $F \in \mathcal{F}$, $\exists m \in \omega$ $\exists K_1, \ldots, K_m \in \mathcal{K}$ SUCH THAT $K_1 \cap \cdots \cap K_m \subseteq F$. BY ASSUMPTION $K_1 \cap \cdots \cap K_m \neq \emptyset$. THUS $F \neq \emptyset$, AND SO $\mathcal{F}$ IS PROPER.

3. $\Rightarrow$ ASSUME $|X| \geq 2$. LET $\emptyset \subset Y \subset X$. THEN $[X) \subset [Y) \subset \mathcal{P}(I)$. SO $[X)$ IS NOT AN ULTRAFILTER.

$\Leftarrow$ ASSUME $|X| = 1$. LET $X = \{x\}$. $\forall Y \subseteq I$, $x \in Y$ OR $x \notin Y$ I.E., $x \in Y$ OR $x \in \overline{Y}$ IE $Y \in [X)$ OR $\overline{Y} \in [X)$. SO $[X)$ IS AN ULTRAFILTER SINCE IT IS CLEARLY PROPER.

4. (a) $\forall x \in X$, $\overline{\{x\}} \in \mathcal{C}f$. HENCE $\bigcap \mathcal{C}f = \emptyset$. IF $\mathcal{C}f \subseteq \mathcal{F}$, THEN $\bigcap \mathcal{F} = \bigcap \mathcal{C}f = \emptyset$. SO $\mathcal{F}$ IS NONPRINCIPAL.

(b) WE SHOW FIRST OF ALL THAT IF $\mathcal{F}$ CONTAINS A FINITE SET X THEN $\mathcal{F}$ IS PRINCIPAL: $\bigcap \mathcal{F} = \bigcap \{F \cap X : F \in \mathcal{F}\}$ SINCE X IS FINITE, $\exists F_1, \ldots, F_m \in \mathcal{F}$ SUCH THAT $\bigcap \mathcal{F} = (F_1 \cap X) \cap \cdots \cap (F_m \cap X) \in \mathcal{F}$. SO $\mathcal{F}$ IS PRINCIPAL.

NOW ASSUME $\mathcal{F}$ IS A NONPRINCIPAL ULTRAFILTER. LET $X \in \mathcal{C}f$. $X \in \mathcal{F}$ OR $\overline{X} \in \mathcal{F}$. BUT $\overline{X} \notin \mathcal{F}$ SINCE $\overline{X}$ IS FINITE. SO $X \in \mathcal{F}$. HENCE $\mathcal{C}f \subseteq \mathcal{F}$.

5. SUPPOSE BY WAY OF CONTRADICTION THAT $\left|\prod_{i \in I} A_i / \Phi(\mathcal{U})\right| > n$. LET $\vec{a}_1, \ldots, \vec{a}_{n+1} \in \prod_{i \in I} A_i$ SUCH THAT THE $\vec{a}_1/\Phi(\mathcal{U}), \ldots, \vec{a}_{n-1}/\Phi(\mathcal{U})$ ARE PAIRWISE DISTINCT. THEN, $\forall i, j \leq n+1$, $i \neq j$, $EQ(\vec{a}_i, \vec{a}_j) \notin \mathcal{U}$ AND HENCE $\overline{EQ}(\vec{a}_i, \vec{a}_j) \in \mathcal{U}$. SO $\bigcap_{\substack{i,j \leq n+1 \\ i \neq j}} \overline{EQ}(\vec{a}_i, \vec{a}_j) \in \mathcal{U}$.

IN PARTICULAR, $\bigcap \overline{EQ}(\vec{a}_i, \vec{a}_j) \neq \emptyset$.

LET $h \in \bigcap \overline{EQ}(\vec{a}_i, \vec{a}_j)$. THEN $a_1(h), a_2(h), \ldots, a_{n+1}(h)$ ARE PAIRWISE DISTINCT. HENCE $|A_h| > N$, A CONTRADICTION.