

Corrected Mat. 1, 2002

1. Let V be any variety. Prove that for any cardinal κ , the free algebra of V with κ free generators is isomorphic to the coproduct of κ copies of the free algebra with one free generator, i.e., $\mathbf{Fr}_\kappa(V) \cong \coprod_{\xi < \kappa} \mathbf{F}_\xi$, where $\mathbf{F}_\xi = \mathbf{Fr}_1(V)$ for each $\xi < \kappa$.

[note: This generalizes the well known result that every free Abelian group is isomorphic to a direct sum of \mathbf{Z} .]

2. Let $\mathcal{K} \subseteq \mathcal{P}(I)$. Then \mathcal{K} is included in a proper filter and hence an ultrafilter iff, for all $n \in \omega$ and all $K_1, \dots, K_n \in \mathcal{K}$, $K_1 \cap \dots \cap K_n \neq \emptyset$.

3. Prove that a principal filter $[X]$ on a set I is an ultrafilter iff $|X| = 1$.

4. Let I be an infinite set, and let \mathcal{F} be a proper filter on I . Prove the following.

(a) If $\mathcal{Cf} \subseteq \mathcal{F}$, then \mathcal{F} is nonprincipal.

(b) If \mathcal{F} is a nonprincipal ultrafilter, then $\mathcal{Cf} \subseteq \mathcal{F}$.

[Note: The condition that \mathcal{F} is an ultrafilter is essential in part (b). Here is a counterexample. Let $I = \omega$ and let $\mathcal{F} = \{X \in \mathcal{Cf} : 0 \in X\}$. Clearly $\mathcal{Cf} \not\subseteq \mathcal{F}$, but it is easy to see that \mathcal{F} is proper and nonprincipal.]

5. Let $N \in \omega$ and let $\langle \mathbf{A}_i : i \in I \rangle$ be a system of Σ -algebras whose cardinalities are bounded above by N , i.e., $|A_i| \leq N$ for all $i \in I$. Let \mathcal{U} be an ultrafilter on I . Prove that $|\prod_{i \in I} \mathbf{A}_i / \Phi(\mathcal{U})| \leq N$.

[Hint: You can use Problem 4, part (b).]