

A set  $E$  of  $\Sigma$ -equations is *consistent* if  $\text{Mod}(E)$  contains a nontrivial algebra.

Let  $X \subseteq \omega \setminus \{0\}$ . Let  $E$  consist of the five laws of groups (of type II) (i.e.  $x \cdot (y \cdot z) \approx (x \cdot y) \cdot z$ ,  $x \cdot e \approx x$ ,  $e \cdot x \approx x$ ,  $x \cdot x^{-1} \approx e$ ,  $x^{-1} \cdot x \approx e$ ), together with the law  $x^n \approx e$  for each  $n \in X$ .  $E$  is consistent iff  $\text{GCD}(X) > 1$ . Indeed, let  $n = \text{GCD}(X)$ . Then  $\mathbf{A}$  is a model of  $E$  iff  $\mathbf{A}$  is a group and every element of  $\mathbf{A}$  is of finite order dividing  $n$ . Thus  $\mathbf{Z}_n$  is a nontrivial model of  $E$  if  $n > 1$ , and the trivial group is the only model of  $E$  if  $n = 1$ . Clearly  $\text{GCD}(X) = 1$  iff  $\text{GCD}(X') = 1$  for some finite  $X' \subseteq X$ . Thus

$E$  is inconsistent iff some finite subset of  $E$  is inconsistent.

This result might appear to depend on special properties of the groups and of the ring of integers, but in fact it is but special case of a much more general result.

**Theorem 3.39** (Compactness Theorem of Equational Logic). *A set of  $\Sigma$ -equations is consistent iff every finite subset is consistent.*

*Proof.* The implication from left to right is trivial. For the opposite implication, assume  $E$  is a set of equations such that every finite subset of  $E$  is consistent. We also assume  $E$  is infinite, since otherwise it is trivially consistent. Let  $\mathcal{P}_\omega(E)$  be the set of all finite subsets of  $E$ . Then by assumption each  $E' \in \mathcal{P}_\omega(E)$  has a nontrivial model, say  $\mathbf{A}_{E'}$ . Consider the  $\mathcal{P}_\omega(E)$ -indexed system of  $\Sigma$ -algebras  $\langle \mathbf{A}_{E'} : E' \in \mathcal{P}_\omega(E) \rangle$  and their direct product  $\prod_{E' \in \mathcal{P}_\omega(E)} \mathbf{A}_{E'}$ . Notice that, for each  $\varepsilon \in E$ ,  $\mathbf{A}_{E'}$  is a model of  $\varepsilon$  for every  $E' \in \mathcal{P}_\omega(E)$  such that  $\varepsilon \in E'$ . We will see that there is a proper filter  $\mathcal{F}$  on  $\mathcal{P}_\omega(E)$  (and hence a subset of  $\mathcal{P}(\mathcal{P}_\omega(E))$ ) such that the set of all such  $E'$  is a member of  $\mathcal{F}$ . Thus by Lem. 3.38 the reduced product  $(\prod_{E' \in \mathcal{P}_\omega(E)} \mathbf{A}_{E'}) / \Phi(\mathcal{F})$  is a model of every  $\varepsilon$  in  $E$ . The construction of  $\mathcal{F}$  is straightforward, but is complicated by the fact that the index set is a set of sets (in fact a set of finite sets) rather than a simple set.

For each  $E' \in \mathcal{P}_\omega(E)$ , let  $[E']$  be the set of all finite subsets of  $E$  that include  $E'$ , i.e.,

$$[E'] = \{ F : E' \subseteq F \in \mathcal{P}_\omega(E) \} \in \mathcal{P}(\mathcal{P}_\omega(E)).$$

$[E']$  is the principal filter generated by  $E'$  in the lattice  $\langle \mathcal{P}_\omega(E), \cup, \cap \rangle$  of finite subsets of  $E$ . (Although it plays no role in our proof, we note that this lattice is not complete because it has no upper bound. We also note that although  $[E']$  consists of finite sets it is itself infinite.) Let  $\mathcal{K} = \{ [E'] : E' \in \mathcal{P}_\omega(E) \}$ . Consider any finite set  $[E'_1], \dots, [E'_n]$  of elements of  $\mathcal{K}$ .  $E'_i \subseteq E'_1 \cup \dots \cup E'_n$  for each  $i \leq n$ . Thus  $E'_1 \cup \dots \cup E'_n \in [E'_1] \cap \dots \cap [E'_n]$  since  $E'_1 \cup \dots \cup E'_n$  is finite. So  $[E'_1] \cap \dots \cap [E'_n]$  is nonempty, and hence  $\mathcal{K}$  has the finite intersection property. Consequently, by Cor. 3.33,  $\mathcal{K}$  is included in a proper filter  $\mathcal{F}$ . Note that both  $\mathcal{K}$  and  $\mathcal{F}$  are subsets of  $\mathcal{P}(\mathcal{P}_\omega(E))$ . Let  $\mathbf{B} = (\prod_{E' \in \mathcal{P}_\omega(E)} \mathbf{A}_{E'}) / \Phi(\mathcal{F})$ . For each  $\varepsilon \in E$ , we have

$$\{ E' \in \mathcal{P}_\omega(E) : \mathbf{A}_{E'} \models \varepsilon \} \supseteq \{ E' \in \mathcal{P}_\omega(E) : \varepsilon \in E' \} = [\{\varepsilon\}] \in \mathcal{F}.$$

So  $\mathbf{B} \models \varepsilon$  by Lem. 3.37. Thus  $\mathbf{B} \in \text{Mod}(E)$ .

It remains only to show that  $\mathbf{B}$  is nontrivial. For every  $E' \in \mathcal{P}_\omega(E)$  choose  $a_{E'}$  and  $b_{E'}$  to be distinct elements of  $\mathbf{A}_{E'}$ ; this is possible since all the  $\mathbf{A}_{E'}$  are nontrivial. Let  $\vec{a} = \langle a_{E'} : E' \in \mathcal{P}_\omega(E) \rangle$  and  $\vec{b} = \langle b_{E'} : E' \in \mathcal{P}_\omega(E) \rangle$ . Then  $\text{EQ}(\vec{a}, \vec{b}) = \emptyset \notin \mathcal{F}$ . So  $\vec{a} / \Phi(\mathcal{F}) \neq \vec{b} / \Phi(\mathcal{F})$ . Hence  $\mathbf{B}$  is nontrivial.  $\square$

We now give another application of the reduced product by showing that every algebra is isomorphic to a subalgebra of a reduced product of its finitely generated subalgebras.

**Theorem 3.40.** *Let  $\mathbf{A}$  be a  $\Sigma$ -algebra. Then  $\mathbf{A} \in \mathbf{SP}_U\{\mathbf{B} : \mathbf{B} \subseteq \mathbf{A}, \mathbf{B} \text{ is finitely generated}\}$ .*

*Proof.* Let  $I = \mathcal{P}_\omega(A)$ . As in the proof of the Compactness Theorem there exists a proper filter  $\mathcal{F}$  on  $\mathcal{P}_\omega(A)$  such that, for every  $X \in \mathcal{P}_\omega(A)$ ,  $[X] = \{Y \in \mathcal{P}_\omega(A) : X \subseteq Y\} \in \mathcal{F}$ .

For each  $X \in \mathcal{P}_\omega(A)$ , let  $\mathbf{B}_X$  be the subalgebra of  $\mathbf{A}$  generated by  $X$ , i.e., the subalgebra with universe  $\text{Sg}^{\mathbf{A}}(X)$ , provided this subuniverse is nonempty. (Of course,  $\text{Sg}^{\mathbf{A}}(X)$  can be empty only if  $X$  is empty.) If it is, take  $\mathbf{B}_\emptyset$  to be any fixed but arbitrary (nonempty) finitely generated subalgebra of  $\mathbf{A}$ . For each  $X \in \mathcal{P}_\omega(A)$  choose a fixed but arbitrary element  $b_X$  of  $\mathbf{B}_X$ . Let  $\mathbf{B} = \prod_{X \in \mathcal{P}_\omega(A)} \mathbf{B}_X$  and consider the mappings

$$A \xrightarrow{h} B \xrightarrow{\Delta_{\Phi(\mathcal{F})}} B/\Phi(\mathcal{F}),$$

where, for each  $a \in A$ ,

$$h(a) = \langle \hat{a}_X : X \in \mathcal{P}_\omega(A) \rangle \quad \text{with} \quad \begin{cases} \hat{a}_X = a & \text{if } a \in B_X, \\ \hat{a}_X = b_X & \text{if } a \notin B_X. \end{cases}$$

Let  $g = \Delta_{\Phi(\mathcal{F})} \circ h$ .  $h$  is not a homomorphism from  $\mathbf{A}$  into  $\mathbf{B}$  (exercise), but we claim that

$$g \text{ is a homomorphism from } \mathbf{A} \text{ into } \mathbf{B}/\Phi(\mathcal{F}).$$

To see this assume  $\sigma \in \Sigma_n$  and let  $a_1, \dots, a_n \in A$ . By the definition of  $h$

$$h(\sigma^{\mathbf{A}}(a_1, \dots, a_n)) = \langle \widehat{\sigma^{\mathbf{A}}(a_1, \dots, a_n)}_X : X \in \mathcal{P}_\omega(A) \rangle,$$

and by the definition of the direct product,

$$\begin{aligned} \sigma^{\mathbf{B}}(h(a_1), \dots, h(a_n)) &= \sigma^{\mathbf{B}}(\langle \hat{a}_{1X} : X \in \mathcal{P}_\omega(A) \rangle, \dots, \langle \hat{a}_{nX} : X \in \mathcal{P}_\omega(A) \rangle) \\ &= \langle \sigma^{\mathbf{B}_X}(\hat{a}_{1X}, \dots, \hat{a}_{nX}) : X \in \mathcal{P}_\omega(A) \rangle. \end{aligned}$$

We now observe that, for every  $X \in \mathcal{P}_\omega(A)$  such that  $a_1, \dots, a_n \in X$  we have  $\sigma^{\mathbf{A}}(a_1, \dots, a_n) \in B_X$  (since  $\mathbf{B}_X$  is a subalgebra of  $\mathbf{A}$ ), and hence

$$\widehat{\sigma^{\mathbf{A}}(a_1, \dots, a_n)}_X = \sigma^{\mathbf{A}}(a_1, \dots, a_n) = \sigma^{\mathbf{B}_X}(a_1, \dots, a_n) = \sigma^{\mathbf{B}_X}(\hat{a}_{1X}, \dots, \hat{a}_{nX}).$$

Thus, for each  $X$  such that  $a_1, \dots, a_n \in X$ ,  $X \in \text{EQ}(h(\sigma^{\mathbf{A}}(a_1, \dots, a_n)), \sigma^{\mathbf{B}}(h(a_1), \dots, h(a_n)))$ . Hence

$$\text{EQ}(h(\sigma^{\mathbf{A}}(a_1, \dots, a_n)), \sigma^{\mathbf{B}}(h(a_1), \dots, h(a_n))) \supseteq [\{a_1, \dots, a_n\}] \in \mathcal{F}.$$

So

$$\begin{aligned} g(\sigma^{\mathbf{A}}(a_1, \dots, a_n)) &= h(\sigma^{\mathbf{A}}(a_1, \dots, a_n))/\Phi(\mathcal{F}) \\ &= \sigma^{\mathbf{B}}(h(a_1), \dots, h(a_n))/\Phi(\mathcal{F}) \\ &= \sigma^{\mathbf{B}/\Phi(\mathcal{F})}(h(a_1)/\Phi(\mathcal{F}), \dots, h(a_n)/\Phi(\mathcal{F})) \\ &= \sigma^{\mathbf{B}/\Phi(\mathcal{F})}(g(a_1), \dots, g(a_n)). \end{aligned}$$

Thus  $g \in \text{Hom}(\mathbf{A}, \mathbf{B}/\Phi(\mathcal{F}))$

We further claim that  $g$  is injective. To see this let  $a$  and  $a'$  be distinct elements of  $A$ . For every  $X \in \mathcal{P}_\omega(A)$  such that  $a, a' \in X$  we have  $\hat{a}_X = a \neq a' = \hat{a}'_X$ , and hence  $X \notin \text{EQ}(h(a), h(a'))$ . Thus  $\text{EQ}(h(a), h(a')) \subseteq [\{a, a'\}]$ . But  $[\{a, a'\}] \notin \mathcal{F}$  since  $[\{a, a'\}] \in \mathcal{F}$  and  $\mathcal{F}$  is proper. So  $\text{EQ}(h(a), h(a')) \notin \mathcal{F}$  and thus  $\langle h(a), h(a') \rangle \notin \Phi(\mathcal{F})$  by definition of  $\Phi(\mathcal{F})$ . Hence  $g(a) \neq g(a')$ . Thus  $g$  is injective as claimed.

We have shown that  $g: \mathbf{A} \rightarrow (\prod_{X \in \mathcal{P}_\omega(A)} \mathbf{B}_X) / \Phi(\mathcal{F})$ . So

$$\mathbf{A} \in \mathbf{SP}_\cup \{ \mathbf{B} : \mathbf{B} \subseteq \mathbf{A}, \mathbf{B} \text{ is finitely generated} \}.$$

□

*Example:* Let  $\mathbf{A} = \langle \mathbb{Z}, S \rangle$ , where  $S$  is the successor function.  $\mathbf{A}$  is not finitely generated. Every finitely generated subalgebra of  $\mathbf{A}$  is of the form  $\mathbf{A}_n = \langle [n] : S \upharpoonright [n] \rangle$ , where  $[n] = \{ k \in \mathbb{Z} : n \leq k \}$ , for each  $n \in \mathbb{Z}$ . The general construction of the reduced product in the proof of the above theorem can be simplified in this special case because every finitely generated subalgebra is in fact generated by a single element. Because of this the index set can be taken to be the set  $\{ \{n\} : n \in \mathbb{Z} \}$  of singleton subsets of  $\mathbb{Z}$  rather than the set of all finite subsets. And then the natural bijection between a set and its corresponding set of singleton subsets allows the further simplification of taking the index set to be  $\mathbb{Z}$  itself.

Thus we take  $I$  to be  $\mathbb{Z}$ , and let  $\mathcal{K} = \{ [n] : n \in \mathbb{Z} \}$ , where  $[n] = \{ k \in \mathbb{Z} : k \leq n \}$ .  $\text{Min}(n_1, \dots, n_k) \in [n_1] \cap \dots \cap [n_k]$  for every finite set  $n_1, \dots, n_k$  of elements of  $\mathbb{Z}$ . So  $\mathcal{K}$  has the finite intersection property, i.e., the intersection of every finite subset of members of  $\mathcal{K}$  is nonempty. Thus by Cor. 3.33  $\mathcal{K}$  is included in a proper filter  $\mathcal{F}$ . We can take  $\mathcal{F}$  to be the smallest such filter, i.e., the filter generated by  $\mathcal{K}$ , which by Lem. 3.31 takes the form  $\mathcal{F} = \{ X : \exists n \in \omega (([n] \subseteq X) \}$ .

Define  $h: A \rightarrow \prod_{n \in \mathbb{Z}} A_n$  such that, for each  $a \in \mathbb{Z}$ ,  $h(a) = \langle \hat{a}_n : n \in \mathbb{Z} \rangle$ , where

$$\hat{a}_n = \begin{cases} a & \text{if } a \in [n] \\ n & \text{otherwise.} \end{cases}$$

Finally, define  $g: A \rightarrow (\prod_{n \in \mathbb{Z}} A_n) / \Phi(\mathcal{F})$  by  $g(a) = h(a) / \Phi(\mathcal{F})$  for each  $a \in \mathbb{Z}$ . See Figure 21.

$h(0)$  and  $h(1)$  are illustrated in Figure 21 by bold dashed and dot-dashed lines, respectively. Note that  $h(0)$  and  $h(1)$  agree on  $[1]$  and disagree on  $(0]$ . Both sets are infinite but only  $(0]$  is in  $\mathcal{F}$ , so  $h(0) / \Phi(\mathcal{F}) \neq h(1) / \Phi(\mathcal{F})$ . But notice that these elements are distinct because  $[1]$  fails to be in the filter, not because  $(0]$  is in the filter.

*Exercises:*

- (1) Show that  $h$  is not a homomorphism from  $\mathbf{A}$  to  $\prod_{n \in \mathbb{Z}} \mathbf{A}_n$ .
- (2) Show that  $g$  is an injective homomorphism from  $\mathbf{A}$  to  $(\prod_{n \in \mathbb{Z}} \mathbf{A}_n) / \Phi(\mathcal{F})$ .
- (3) Note that each  $\mathbf{A}_n$  is isomorphic to  $\langle \omega, S \rangle$ ; thus  $\langle \mathbb{Z}, S \rangle$  is isomorphic to a subalgebra of a reduced power of  $\langle \omega, S \rangle$ . Show that a reduced product is really necessary here by proving that  $\langle \mathbb{Z}, S \rangle$  is not isomorphic to a subalgebra of any power of  $\langle \omega, S \rangle$ , i.e.  $\langle \mathbb{Z}, S \rangle \notin \mathbf{SP}(\{ \langle \omega, S \rangle \})$ .

It sufficed in the proof of the Compactness Theorem to use a proper filter rather than an ultrafilter because we were only interested in preserving identities. If we want to preserve nonidentities we have to use ultrafilters. This fact is reflected in the following lemma.

**Lemma 3.41.** *Let  $\langle \mathbf{A}_i : i \in I \rangle$  be a system of  $\Sigma$ -algebras and let  $\mathcal{F}$  be filter on  $I$ . Let  $\varepsilon$  be a  $\Sigma$ -equation. If  $\mathcal{F}$  is an ultrafilter, then*

$$\left( \prod_{i \in I} \mathbf{A}_i \right) / \Phi(\mathcal{F}) \neq \varepsilon \quad \text{iff} \quad \{ i \in I : \mathbf{A}_i \neq \varepsilon \} \in \mathcal{F}.$$

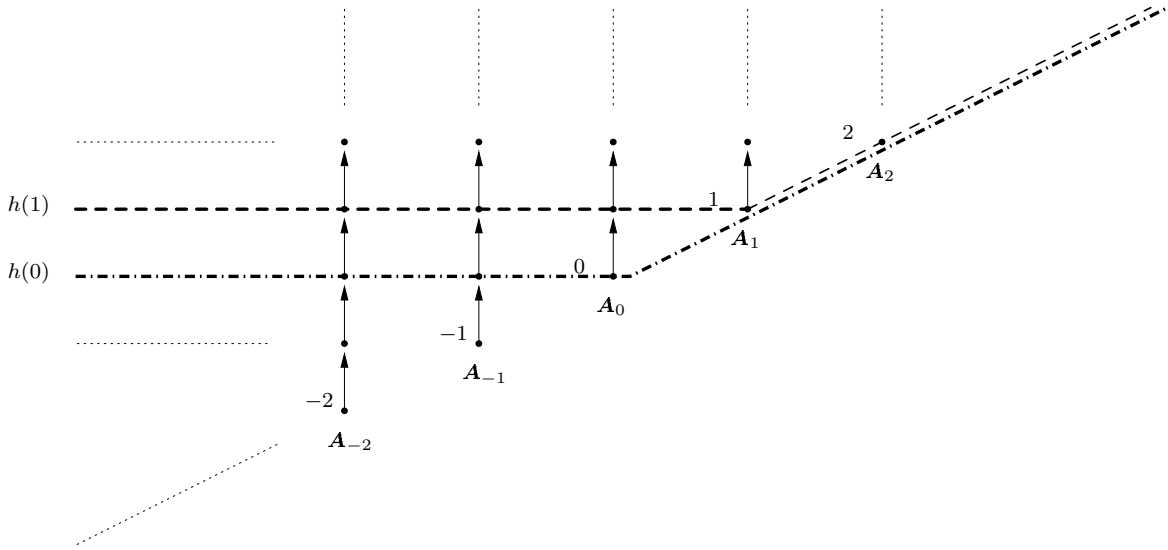


FIGURE 21

*Proof.*

$$\begin{aligned}
 \left(\prod_{i \in I} \mathbf{A}_i\right) / \Phi(\mathcal{F}) \not\models \varepsilon & \text{ iff } \{i \in I : \mathbf{A}_i \models \varepsilon\} \notin \mathcal{F}, \text{ by Lem. 3.38} \\
 & \text{ iff } \overline{\{i \in I : \mathbf{A}_i \models \varepsilon\}} \in \mathcal{F}, \text{ by Thm. 3.34} \\
 & \text{ iff } \{i \in I : \mathbf{A}_i \not\models \varepsilon\} \in \mathcal{F}.
 \end{aligned}$$

□

That  $\mathcal{F}$  is an ultrafilter is necessary here. Let  $\varepsilon$  be any  $\Sigma$ -equation, and let  $\mathbf{B}$  and  $\mathbf{C}$  be  $\Sigma$ -algebras such that  $\mathbf{B} \not\models \varepsilon$  and  $\mathbf{C} \models \varepsilon$ . For each  $n \in \omega$  let  $\mathbf{A}_n = \mathbf{B}$  if  $n$  is even and  $\mathbf{A}_n = \mathbf{C}$  if  $n$  is odd. Let  $\mathcal{F} = \mathcal{C}f$ , the filter of cofinite subsets of  $\omega$ .  $\{n \in \omega : \mathbf{A}_n \models \varepsilon\} = \{2n + 1 : n \in \omega\} \notin \mathcal{F}$ , so  $(\prod_{n \in \omega} \mathbf{A}_n) / \Phi(\mathcal{C}f) \not\models \varepsilon$  by Lem. 3.38. On the other hand,  $\{n \in \omega : \mathbf{A}_n \not\models \varepsilon\} = \{2n : n \in \omega\} \notin \mathcal{F}$ .

Ultraproducts also preserve more logically complex conditions. We give an example which depends on the following result about ultrafilters.

**Lemma 3.42.** *Let  $\mathcal{U}$  be an ultrafilter on a set  $I$ .*

- (i) *Let  $X_1, \dots, X_n$  be any finite set of subsets of  $I$ . Then  $X_1 \cup \dots \cup X_n \in \mathcal{U}$  iff  $X_i \in \mathcal{U}$  for at least one  $i \leq n$ .*
- (ii) *Let  $X_1, \dots, X_n$  and  $Y_1, \dots, Y_m$  be two finite sets of subsets of  $I$ . Then  $X_1 \cup \dots \cup X_n \cup \overline{Y_1} \cup \dots \cup \overline{Y_m} \in \mathcal{U}$  iff either some  $X_i \in \mathcal{U}$  or some  $Y_j \notin \mathcal{U}$ .*

The proof is left as an exercise. Hint: prove the first part by induction on  $n$ .

**Definition 3.43.** Let

$$\varepsilon_1 = t_1(x_0, \dots, x_{k-1}) \approx s_1(x_0, \dots, x_{k-1}), \dots, \varepsilon_n = t_n(x_0, \dots, x_{k-1}) \approx s_n(x_0, \dots, x_{k-1})$$

and

$$\delta_1 = u_1(x_0, \dots, x_{k-1}) \approx v_1(x_0, \dots, x_{k-1}), \dots, \delta_m = u_m(x_0, \dots, x_{k-1}) \approx v_m(x_0, \dots, x_{k-1})$$

be two finite sets of  $\Sigma$ -equations. A  $\Sigma$ -algebra  $\mathbf{A}$  is said to be a model of the formula

$$\varepsilon_1 \text{ or } \dots \text{ or } \varepsilon_n \text{ or } (\text{not } \delta_1) \text{ or } \dots \text{ or } (\text{not } \delta_m),$$

which in turn is said to be *universally valid in  $\mathbf{A}$* , if, for all  $a_0, \dots, a_{k-1} \in A$ , either  $t_i^{\mathbf{A}}(a_0, \dots, a_{k-1}) = s_i^{\mathbf{A}}(a_0, \dots, a_{k-1})$  for some  $i \leq n$  or  $u_j^{\mathbf{A}}(a_0, \dots, a_{k-1}) \neq v_j^{\mathbf{A}}(a_0, \dots, a_{k-1})$  for some  $j \leq m$ . In this case we write

$$\mathbf{A} \models \varepsilon_1 \text{ or } \dots \text{ or } \varepsilon_n \text{ or } (\text{not } \delta_1) \text{ or } \dots \text{ or } (\text{not } \delta_m).$$

By a  $(\Sigma)$ -*equational literal* we mean either a  $\Sigma$ -equation or a formula of the form **not**  $\varepsilon$  where  $\varepsilon$  is a  $\Sigma$ -equation. A finite disjunction of equational literals, such as  $\varepsilon_1 \text{ or } \dots \text{ or } \varepsilon_n \text{ or } (\text{not } \delta_1) \text{ or } \dots \text{ or } (\text{not } \delta_m)$ , is called a  $(\Sigma)$ -*equational clause*. Strictly speaking this formula should be written in the form

$$\forall x_0 \dots \forall x_{k-1} (\varepsilon_1 \text{ or } \dots \text{ or } \varepsilon_n \text{ or } (\text{not } \delta_1) \text{ or } \dots \text{ or } (\text{not } \delta_m)),$$

with universal quantifiers at the front, but they are normally omitted for simplicity. The classes of all models of an equational clause  $\psi$ , and of a set  $\Psi$  of equational clauses, are written respectively as  $\text{Mod}(\psi)$  and  $\text{Mod}(\Psi)$ .

Many important properties of algebras can be expressed as equational clauses. For example the property of a commutative ring with identity that it be an integral domain can be expressed as

$$(x \approx 0) \text{ or } (y \approx 0) \text{ or } ((\text{not}(x \cdot y \approx 0)).$$

We have the following generalization of Lemma 3.38 when applied to ultrafilters.

**Lemma 3.44.** *Let  $\langle \mathbf{A}_i : i \in I \rangle$  be a system of  $\Sigma$ -algebras, and let  $\mathcal{U}$  be an ultrafilter on  $I$ . Let  $\psi$  be an equational clause. Then*

$$\left( \prod_{i \in I} \mathbf{A}_i \right) / \Phi(\mathcal{U}) \models \psi \quad \text{iff} \quad \{ i \in I : \mathbf{A}_i \models \psi \} \in \mathcal{U}.$$

The proof is also left as an exercise. It is similar to that of Lemma 3.38, but of course it uses Lemma 3.42.

The following theorem is an immediate consequence of the last lemma.

**Theorem 3.45.** *Let  $\Psi$  be a set of equational clauses. Then  $\mathbf{P}_U(\text{Mod}(\Psi)) = \text{Mod}(\Psi)$ .*

So the class of all integral domains is closed under taking ultraproducts. This is not true of course of arbitrary reduced products. The ring of integers  $\mathbf{Z}$  is an integral domain but its square  $\mathbf{Z} \times \mathbf{Z}$  is not ( $\langle 0, 1 \rangle$  is a zero divisor). Note that every direct product is a reduced product; more precisely,  $\prod_{i \in I} \mathbf{A}_i \cong (\prod_{i \in I} \mathbf{A}_i) / \Phi([I])$ .