3.3. **Reduced Products and Ultraproducts.** Let I be a nonempty set. Let

$$\boldsymbol{\mathcal{P}}(I) = \langle \mathcal{P}(I), \cup, \cap, \overline{}, \emptyset, I \rangle,$$

where, for every $X \subseteq I$, $\overline{X} = I \setminus X$ is the *complement* of X relative to I. $\boldsymbol{\mathcal{P}}(I)$ is the Boolean algebra of all subsets of I. $\mathcal{F} \subseteq \mathcal{P}(I)$ is a *filter on* or *over* I if $\mathcal{F}$ is a dual ideal of the lattice $\langle \mathcal{P}(I), \cup, \cap, \emptyset, I \rangle$, i.e.,

(i) $\mathcal{F}$ is nonempty;
(ii) $\mathcal{F}$ is an upper segment, i.e., $X \in \mathcal{F}$ and $X \subseteq Y$ implies $Y \in \mathcal{F}$;
(iii) $\mathcal{F}$ is closed under intersection, i.e., $X, Y \in \mathcal{F}$ implies $X \cap Y \in \mathcal{F}$.

The set of filters of I is an algebraic closed-set system, since the set of ideals of any lattice forms one. Because of (ii), the condition (i) is equivalent to $I \in \mathcal{F}$. A filter $\mathcal{F}$ is *proper* if $\mathcal{F} \neq \mathcal{P}(I)$. Because of (ii), $\mathcal{F}$ is proper iff $\emptyset \notin \mathcal{F}$. Thus the union of any chain of proper filters is a proper filter, and consequently Zorn's lemma can be applied to show that every proper filter $\mathcal{F}$ is included in a maximal proper filter, that is, a proper filter $\mathcal{U}$ such that there is no filter $\mathcal{G}$ such that $\mathcal{U} \subset \mathcal{G} \subset \mathcal{P}(I)$. Maximal proper filters are called *ultrafilters.*

Examples:

(1) For $J \subseteq I$, $\mathcal{P}(I)[J) = \{\, X : J \subseteq X \subseteq I \,\}$ is the *principal filter generated by* J; for simplicity we normally write $[J)$ for $\mathcal{P}(I)[J)$. A filter $\mathcal{F}$ is principal iff $\bigcap \mathcal{F}\ (= \bigcap\{\, F : F \in \mathcal{F} \,\}) \in \mathcal{F}$, in which case $\mathcal{F} = [\bigcap \mathcal{F})$. Thus, if I is finite, every filter $\mathcal{F}$ on I is principal. The smallest filter is $[I) = \{I\}$ and the largest filter, the improper filter, is $[\emptyset) = \mathcal{P}(I)$.

(2) Every nonprincipal filter must be over an infinite set. A subset X of I is *cofinite* if $\overline{X}$ is finite. Let $\mathcal{C}f$ be the set of all cofinite subsets of I. Clearly I is cofinite, and any superset of a cofinite set is cofinite. If X and Y are cofinite, then $\overline{X \cap Y} = \overline{X} \cup \overline{Y}$ is finite, and hence $\mathcal{C}f$ is closed under intersection. So $\mathcal{C}f$ is a filter. $\emptyset$ is cofinite iff I is finite. So $\mathcal{C}f$ is proper iff I is infinite. For each $i \in I$, $\overline{\{i\}}$ is obviously cofinite. Thus $\bigcap \mathcal{C}f \subseteq \bigcap_{i \in I} \overline{\{i\}} = \emptyset$. Hence $\mathcal{C}f$ is nonprincipal if I is infinite.

Lemma 3.31. *Let I be a set, and let $\mathcal{K}$ be an arbitrary set of subsets of I. Let $\mathcal{F}$ be the filter generated by $\mathcal{K}$, i.e., $\mathcal{F} := \bigcap\big\{\, \mathcal{G} : \mathcal{G}$ a filter such that $\mathcal{K} \subseteq \mathcal{G} \,\big\}$. Then*

$$\mathcal{F} = \big\{\, X : \exists\, n \in \omega\, \exists\, K_1, \dots, K_n \in \mathcal{K}(K_1 \cap \cdots \cap K_n \subseteq X) \,\big\}.$$

Proof. Let $\mathcal{H} = \big\{\, X : \exists\, n \in \omega\, \exists\, K_1, \dots, K_n \in \mathcal{K}(K_1 \cap \cdots \cap K_n \subseteq X) \,\big\}$. If $\mathcal{K}$ is empty, then the only sequence $K_1, \dots, K_n$ of members of $\mathcal{K}$ is the empty sequence (n=0). Then, by definition of the intersection of an empty sequence, $K_1 \cap \cdots \cap K_n = I$. Thus $I \in \mathcal{H}$, and in fact $\mathcal{H} = [I) = \{I\}$, the smallest filter. And $\mathcal{F} = \{I\}$, being in this case the intersection of all filters.

Now suppose $\mathcal{K}$ is nonempty. We first verify that $\mathcal{H}$ is a filter that includes $\mathcal{K}$. For each $K_1 \in \mathcal{K}$, $K_1 \subseteq K_1$, and hence $K_1 \in \mathcal{H}$. Thus $\mathcal{K} \subseteq \mathcal{F}$. If $\mathcal{K}$ is nonempty, so is $\mathcal{H}$. Suppose $X \in \mathcal{H}$; say $K_1 \cap \cdots \cap K_n \subseteq X$ with $K_1, \dots, K_n \in \mathcal{K}$. Then $K_1 \cap \cdots \cap K_n \subseteq Y$, and hence $Y \in \mathcal{H}$, for every Y such that $X \subseteq Y$. So $\mathcal{H}$ is an upper segment. Suppose $X, Y \in \mathcal{H}$. Then $K_1 \cap \cdots \cap K_n \subseteq X$ and $L_1 \cap \cdots \cap L_m \subseteq Y$ with $K_1, \dots, K_n, L_1, \dots, L_m \in \mathcal{K}$. Then $K_1 \cap \cdots \cap K_n \cap L_1 \cap \cdots \cap L_m \subseteq X \cap Y$. So $X \cap Y \in \mathcal{H}$, and hence $\mathcal{H}$ is closed under intersection. Thus $\mathcal{H}$ is a filter.

We have seen that $\mathcal{K} \subseteq \mathcal{H}$. Let $\mathcal{G}$ be a filter such that $\mathcal{K} \subseteq \mathcal{G}$. Then $K_1 \cap \cdots \cap K_n \in \mathcal{G}$ for all $K_1, \ldots, K_n \in \mathcal{K}$, and hence $X \in \mathcal{G}$ for every X such that $K_1 \cap \cdots \cap K_n \subseteq X$, since $\mathcal{G}$ is an upper segment. So $\mathcal{H} \subseteq \mathcal{G}$. Thus $\mathcal{H} = \mathcal{F}$. □

Corollary 3.32. *Let $\mathcal{F}$ is a filter over I, and let $X \in \mathcal{P}(I)$. Let $\mathcal{G}$ be the smallest filter including $\mathcal{F}$ that contains X, i.e., the filter generated by $\mathcal{K} = \mathcal{F} \cup \{X\}$. Then*

$$\mathcal{G} = \{\, Y \subseteq I : \exists\, F \in \mathcal{F}\, (F \cap X \subseteq Y)\,\}.$$

Proof. Let $\mathcal{H} = \{\, Y \subseteq I : \exists\, F \in \mathcal{F}\, (F \cap X \subseteq Y)\,\}$. By the lemma, $\mathcal{G} = \{Y \subseteq I : \exists\, n \in \omega\, \exists\, K_1, \ldots, K_n \in \mathcal{F} \cup \{X\}\, (K_1 \cap \cdots \cap K_n \subseteq Y)\,\}$. Clearly $\mathcal{H} \subseteq \mathcal{G}$. Let $Y \in \mathcal{G}$. Then

$$K_1 \cap \cdots \cap K_n \subseteq Y, \tag{26}$$

for some $K_1, \ldots, K_n \in \mathcal{F} \cup \{X\}$. Suppose $X = K_i$ for some $i \leq n$; without loss of generality assume $X = K_n$. Then

$$K_1 \cap \cdots \cap K_n = \underbrace{K_1 \cap \cdots \cap K_{n-1}}_{F \in \mathcal{F}} \cap X \subseteq Y.$$

So $Y \in \mathcal{H}$. If $X \neq K_i$ for all $i \leq n$, then $K_1 \cap \cdots \cap K_n = F \in \mathcal{F}$, and hence (26) implies $F \cap X \subseteq Y$. So again $Y \in \mathcal{H}$. So $\mathcal{G} \subseteq \mathcal{H}$. □

Corollary 3.33. *Let $\mathcal{K} \subseteq \mathcal{P}(I)$. Then $\mathcal{K}$ is included in a proper filter and hence an ultrafilter iff, for all $n \in \omega$ and all $K_1, \ldots, K_n \in \mathcal{K}$, $K_1 \cap \cdots \cap K_n \neq \emptyset$.*

Proof. Exercise. □

A set $\mathcal{K}$ of subsets of a nonempty set I is said to have the *finite intersection property* if the intersection of every finite subset of $\mathcal{K}$ is nonempty. By the above corollary, every set of subsets of I with this property is included in a proper filter.

The following gives a convenient characterization of ultrafilters.

Theorem 3.34. *Let $\mathcal{F}$ be a filter over a set I. $\mathcal{F}$ is an ultrafilter iff*

$$\text{for every } X \subseteq I, \text{ either } X \in \mathcal{F} \text{ or } \overline{X} \in \mathcal{F}, \text{ but not both.} \tag{27}$$

Proof. $\Longleftarrow$ Assume (27) holds. Then $\emptyset \notin \mathcal{F}$ since $I \in \mathcal{F}$ and $\emptyset = \overline{I}$. So $\mathcal{F}$ is proper. Let $\mathcal{G}$ be a filter such that $\mathcal{F} \subset \mathcal{G}$. Let $X \in \mathcal{G} \setminus \mathcal{F}$. Then by (27) $\overline{X} \in \mathcal{F} \subseteq \mathcal{G}$. Thus $\emptyset = X \cap \overline{X} \in \mathcal{G}$, i.e., $\mathcal{G} = \mathcal{P}(I)$. Thus $\mathcal{F}$ is an ultrafilter.

$\Longrightarrow$ Suppose $\mathcal{F}$ is an ultrafilter and $X \notin \mathcal{F}$. Since $\mathcal{F}$ is maximal and proper, $\mathcal{P}(I)$ is smallest filter including $\mathcal{F}$ that contains X. By Cor. 3.32 $\mathcal{P}(I) = \{\, Y \subseteq I : \exists\, F \in \mathcal{F}\, (F \cap X \subseteq Y)\,\}$. Thus there is an $F \in \mathcal{F}$ such that $F \cap X = \emptyset$. So $F \subseteq \overline{X}$, and hence $\overline{X} \in \mathcal{F}$. □

Exercises:

(1) A principal filter $[X)$ is an ultrafilter iff $|X| = 1$.

The filter $\mathcal{C}f$ of cofinite sets is never an ultrafilter. If I is finite, $\mathcal{C}f$ is the improper filter. If I if infinite, then I includes a set X such that neither X nor $\overline{X}$ is finite, and hence neither X nor $\overline{X}$ is cofinite.

(2) Let I be infinite. Then $\mathcal{C}f$ is the smallest nonprincipal filter on I, i.e., for any filter $\mathcal{F}$ on I, $\mathcal{F}$ is nonprincipal iff $\mathcal{C}f \subseteq \mathcal{F}$.

Let $\langle\, \boldsymbol{A}_i : i \in I \,\rangle$ be a system of Σ-algebras, and let $\mathcal{F}$ be a filter on I. Define $\Phi(\mathcal{F}) \subseteq \big(\prod_{i\in I} A_i\big)^2$ by the condition that

$$\big\langle \underbrace{\langle\, a_i : i \in I \,\rangle}_{\vec{a}}, \underbrace{\langle\, b_i : i \in I \,\rangle}_{\vec{b}} \big\rangle \in \Phi(\mathcal{F}) \quad \text{iff} \quad \underbrace{\{\, i \in I : a_i = b_i \,\}}_{\mathrm{EQ}(\vec{a},\vec{b})} \in \mathcal{F},$$

where $\mathrm{EQ}(\vec{a},\vec{b}) := \{\, i \in I : a_i = b_i \,\}$ is called the *equality set* of $\vec{a}$ and $\vec{b}$. Note that $\langle \vec{a},\vec{b}\rangle \in \Phi(\mathcal{C}f)$ iff $\mathrm{EQ}(\vec{a},\vec{b})$ is cofinite, i.e., iff $\{\, i \in I : a_i \neq b_i \,\}$ is finite. It is traditional to say that $\vec{a}$ and $\vec{b}$ are equal "almost everywhere" in this case.

Lemma 3.35. *$\Phi(\mathcal{F}) \in \mathrm{Co}(\prod_{i\in I} \boldsymbol{A}_i)$ for every filter $\mathcal{F}$ on I.*

Proof. $\mathrm{EQ}(\vec{a},\vec{a}) = I \in \mathcal{F}$. So $\Phi(\mathcal{F})$ is reflexive, and it is symmetric because $\mathrm{EQ}(\vec{a},\vec{b}) = \mathrm{EQ}(\vec{b},\vec{a})$.

$$\underbrace{i \in \mathrm{EQ}(\vec{a},\vec{b})}_{a_i=b_i} \quad \text{and} \quad \underbrace{i \in \mathrm{EQ}(\vec{b},\vec{c})}_{b_i=c_i} \quad \text{implies} \quad \underbrace{i \in \mathrm{EQ}(\vec{a},\vec{c})}_{a_i=c_i}.$$

I.e., $\mathrm{EQ}(\vec{a},\vec{b}) \cap \mathrm{EQ}(\vec{b},\vec{c}) \subseteq \mathrm{EQ}(\vec{a},\vec{c})$. So if $\mathrm{EQ}(\vec{a},\vec{b})$ and $\mathrm{EQ}(\vec{b},\vec{c})$ are both in $\mathcal{F}$, then so is $\mathrm{EQ}(\vec{a},\vec{c})$. This means that $\Phi(\mathcal{F})$ is transitive.

Let $\sigma \in \Sigma_n$ and $\vec{a}_1,\dots,\vec{a}_n,\vec{b}_1,\dots,\vec{b}_n \in \prod_{i\in I} A_i$ such that $\mathrm{EQ}(\vec{a}_1,\vec{b}_1),\dots,\mathrm{EQ}(\vec{a}_n,\vec{b}_n) \in \mathcal{F}$. Then as in the proof of transitivity it can be shown that $\mathrm{EQ}(\vec{a}_1,\vec{b}_1) \cap \cdots \cap \mathrm{EQ}(\vec{a}_n,\vec{b}_n) \subseteq \mathrm{EQ}\big(\sigma^{\prod \boldsymbol{A}_i}(\vec{a}_1,\dots,\vec{a}_n), \sigma^{\prod \boldsymbol{A}_i}(\vec{b}_1,\dots,\vec{b}_n)\big) \in \mathcal{F}$. So $\Phi(\mathcal{F})$ has the substitution property. $\square$

$\Phi(\mathcal{F})$ is called the *filter congruence defined by* $\mathcal{F}$.

Definition 3.36. Let $\vec{\boldsymbol{A}} = \langle\, \boldsymbol{A}_i : i \in I \,\rangle$ be a system of Σ-algebras. A Σ-algebra $\boldsymbol{B}$ is a *reduced product of* $\vec{\boldsymbol{A}}$ if $\boldsymbol{B} = \big(\prod_{i\in I} \boldsymbol{A}_i\big)/\Phi(\mathcal{F})$ for some filter $\mathcal{F}$ on I. $\boldsymbol{B}$ is called an *ultraproduct of* $\vec{\boldsymbol{A}}$ if $\mathcal{F}$ is an ultrafilter.

Note that $\boldsymbol{B} \preccurlyeq \prod_{i\in I} \boldsymbol{A}_i$, i.e., $\boldsymbol{B}$ is a homomorphic image of $\prod_{i\in I} \boldsymbol{A}_i$, but it is a very special kind of homomorphic image as we shall see. For any Σ-algebra $\boldsymbol{C}$, we write $\boldsymbol{C} \preccurlyeq_{\mathrm{R}} \prod_{i\in I} \boldsymbol{A}_i$ if $\boldsymbol{C}$ is isomorphic to a reduced product of $\vec{\boldsymbol{A}}$; by the First Isomorphism Theorem, $\boldsymbol{C} \preccurlyeq_{\mathrm{R}} \prod_{i\in I} \boldsymbol{A}_i$ iff $\boldsymbol{C}$ is a homomorphic image of $\prod_{i\in I} \boldsymbol{A}_i$ by a homomorphism whose relation kernel is a filter congruence. We write $\boldsymbol{C} \preccurlyeq_{\mathrm{U}} \prod_{i\in I} \boldsymbol{A}_i$ if $\boldsymbol{C}$ is isomorphic to a ultraproduct of $\vec{\boldsymbol{A}}$.

For any class K of Σ-algebras,

$$\mathbf{P}_{\mathrm{R}}(\mathsf{K}) := \Big\{\, \boldsymbol{B} : \exists I\, \exists \vec{\boldsymbol{A}} \in \mathsf{K}^I\, (\boldsymbol{B} \preccurlyeq_{\mathrm{R}} \prod_{i\in I} \boldsymbol{A}_i) \,\Big\}.$$

$\mathbf{P}_{\mathrm{U}}(\mathsf{K})$ is similarly defined with "$\preccurlyeq_{\mathrm{U}}$" in place of "$\preccurlyeq_{\mathrm{R}}$".

Let I be a set and $\mathcal{F}$ a filter on I. Let $J \subseteq I$ and define

$$\mathcal{F}{\restriction}J := \{\, F \cap J : F \in \mathcal{F} \,\}.$$

$\mathcal{F}{\restriction}J$ is a filter on J: we verify the three defining properties of a filter.

$J = I \cap J \in \mathcal{F}{\restriction}J$. Suppose $X \in \mathcal{F}{\restriction}J$ and $X \subseteq Y \subseteq J$. Let $F \in \mathcal{F}$ such that $X = F \cap J$. Then $F \cup Y \in \mathcal{F}$ and $Y = X \cup Y = (F \cap J) \cup (Y \cap J) = (F \cup Y) \cap J \in \mathcal{F}{\restriction}J$. Finally, suppose $X, Y \in \mathcal{F}{\restriction}J$, and let $F, G \in \mathcal{F}$ such that $X = F \cap J$ and $Y = G \cap J$. Then $X \cap Y = (F \cap G) \cap J \in \mathcal{F}{\restriction}J$.

It is easy to see that, if $J \in \mathcal{F}$, then

$$\mathcal{F}{\restriction}J = \mathcal{P}(J) \cap \mathcal{F}(= \{\, X \subseteq J : X \in \mathcal{F} \,\}).$$

The inclusion from right to left holds for all $J \subseteq I$, without the assumption that $J \in \mathcal{F}$. Assume, $X \in \mathcal{F}{\restriction}J$, i.e., $X = F \cap J$ for some $F \in \mathcal{F}$. Then $X \in \mathcal{F}$ since $J \in \mathcal{F}$.

The following will prove useful in the sequel.

$$\text{If } J \in \mathcal{F}, \text{ then } \forall\, X \subseteq I (X \in \mathcal{F} \quad \text{iff} \quad X \cap J \in \mathcal{F}{\restriction}J). \tag{28}$$

$X \in \mathcal{F}$ implies $X \cap J \in \mathcal{F}{\restriction}J$ by the definition of $\mathcal{F}{\restriction}J$. For the implication in the other direction, assume $X \cap J \in \mathcal{F}{\restriction}J$. Then, since $J \in \mathcal{F}$, $X \cap J \in \mathcal{F}$ by the above characterization of $\mathcal{F}{\restriction}J$ when $J \in \mathcal{F}$. Hence $X \in \mathcal{F}$ since $\mathcal{F}$ is an upper segment.

Lemma 3.37. *Let $\langle\, \boldsymbol{A}_i : i \in I \,\rangle$ be a system of Σ-algebras and $\mathcal{F}$ a filter on I. Then, for each $J \in \mathcal{F}$,*

$$\Big(\prod_{i\in I} \boldsymbol{A}_i\Big)/\varPhi(\mathcal{F}) \cong \Big(\prod_{j\in J} \boldsymbol{A}_j\Big)/\varPhi(\mathcal{F}{\restriction}J).$$

Proof. Consider the epimorphisms

$$\prod_{i\in I} \boldsymbol{A}_i \overset{\pi_J}{\twoheadrightarrow} \prod_{j\in J} \boldsymbol{A}_j \overset{\varDelta_{\varPhi(\mathcal{F}\restriction J)}}{\twoheadrightarrow} \Big(\prod_{j\in J} \boldsymbol{A}_j\Big)/\varPhi(\mathcal{F}{\restriction}J),$$

where $\pi_J(\underbrace{\langle\, a_i : i \in I \,\rangle}_{\vec{a}}) = \underbrace{\langle\, a_j : j \in J \,\rangle}_{\vec{a}\restriction J}$. π_J is the *J-projection function*, and it is easily checked that it is an epimorphism; it generalizes the ordinary projection function π_i, which can be identified with $\pi_{\{i\}}$. $\varDelta_{\varPhi(\mathcal{F}\restriction J)}$ is of course the natural map.

Let $h = \varDelta_{\varPhi(\mathcal{F}\restriction J)} \circ \pi_J \colon \prod_{i\in I} \boldsymbol{A}_i \twoheadrightarrow \big(\prod_{j\in J} \boldsymbol{A}_j\big)/\varPhi(\mathcal{F}{\restriction}J)$. Let $\vec{a} = \langle\, a_i : i \in I \,\rangle$ and $\vec{b} = \langle\, b_i : i \in I \,\rangle$.

$$\begin{aligned}
\langle \vec{a}, \vec{b} \rangle \in \operatorname{rker}(h) \quad &\text{iff} \quad \varDelta_{\varPhi(\mathcal{F}\restriction J)}(\vec{a}{\restriction}J) = \varDelta_{\varPhi(\mathcal{F}\restriction J)}(\vec{b}{\restriction}J)\\
&\text{iff} \quad \langle \vec{a}{\restriction}J, \vec{b}{\restriction}J \rangle \in \varPhi(\mathcal{F}{\restriction}J)\\
&\text{iff} \quad \mathrm{EQ}(\vec{a}{\restriction}J, \vec{b}{\restriction}J)\,(= \{\, j \in J : a_j = b_j \,\}) \in \mathcal{F}{\restriction}J\\
&\text{iff} \quad \mathrm{EQ}(\vec{a}, \vec{b})\,(= \{\, i \in I : a_i = b_i \,\}) \in \mathcal{F};
\end{aligned}$$

this last equivalence holds by (28) since $\mathrm{EQ}(\vec{a}{\restriction}J, \vec{b}{\restriction}J) = \mathrm{EQ}(\vec{a}, \vec{b}) \cap J$

$$\text{iff} \quad \langle \vec{a}, \vec{b} \rangle \in \varPhi(\mathcal{F}).$$

So $\operatorname{rker}(h) = \varPhi(\mathcal{F})$. Now apply the First Isomorphism Theorem. □

By the next lemma, a product $\prod_{i\in I} \boldsymbol{A}_i$ that is reduced by the filter congruence defined by the filter of cofinite sets is a model of a given identity iff the factor $\boldsymbol{A}_i$ is a model of the identity for "almost all" i.

Lemma 3.38. *Let $\langle\, \boldsymbol{A}_i : i \in I \,\rangle$ be a system of Σ-algebras, and let $\mathcal{F}$ be a filter on I. Let ε be an arbitrary Σ-equation. Then*

$$\Big(\prod_{i\in I} \boldsymbol{A}_i\Big)/\varPhi(\mathcal{F}) \vDash \varepsilon \quad \textit{iff} \quad \{\, i \in I : \boldsymbol{A}_i \vDash \varepsilon \,\} \in \mathcal{F}.$$

Proof. Let $J = \{\, i \in I : \boldsymbol{A}_i \vDash \varepsilon \,\}$.

$\Longleftarrow$ Assume $J \in \mathcal{F}$. Then by Lem. 3.37, $\big(\prod_{i\in I} \boldsymbol{A}_i\big)/\mathit{\Phi}(\mathcal{F}) \cong \big(\prod_{j\in J} \boldsymbol{A}_j\big)/\mathit{\Phi}(\mathcal{F}{\restriction}J) \in \mathbf{H}\,\mathbf{P}(\{\, \boldsymbol{A}_j : j \in J \,\}) \subseteq \mathbf{H}\,\mathbf{P}(\mathsf{Mod}(\varepsilon)) = \mathsf{Mod}(\varepsilon)$; the last equality holds by Thm. 3.20. Thus $\big(\prod_{i\in I} \boldsymbol{A}_i\big)/\mathit{\Phi}(\mathcal{F}) \vDash \varepsilon$.

$\Longrightarrow$ Suppose $J \notin \mathcal{F}$. Let $\varepsilon = t(x_0, \dots, x_{n-1}) \approx s(x_0, \dots, x_{n-1})$. For each $i \in I \setminus J$, choose $a_0(i), \dots, a_{n-1}(i) \in A_i$ such that $t^{\boldsymbol{A}_i}\big(a_0(i), \dots, a_{n-1}(i)\big) \neq s^{\boldsymbol{A}_i}\big(a_0(i), \dots, a_{n-1}(i)\big)$. This is possible since $\boldsymbol{A}_i \nvDash \varepsilon$. For each $i \in J$, let $a_0(i), \dots, a_{n-1}(i)$ be arbitrary elements of $\boldsymbol{A}_i$. Let $\vec{a}_0 = \langle\, \vec{a}_0(i) : i \in I \,\rangle, \dots, \vec{a}_{n-1} = \langle\, \vec{a}_{n-1}(i) : i \in I \,\rangle$. Recall, that

$$t^{\prod \boldsymbol{A}_i}(\vec{a}_0, \dots, \vec{a}_{n-1}) = \big\langle\, t^{\boldsymbol{A}_i}\big(\vec{a}_0(i), \cdots, \vec{a}_{n-1}(i)\big) : i \in I \,\big\rangle \text{ and}$$
$$s^{\prod \boldsymbol{A}_i}(\vec{a}_0, \dots, \vec{a}_{n-1}) = \big\langle\, s^{\boldsymbol{A}_i}\big(\vec{a}_0(i), \cdots, \vec{a}_{n-1}(i)\big) : i \in I \,\big\rangle.$$

Thus

$$\begin{aligned}
&\mathrm{EQ}\big(t^{\prod \boldsymbol{A}_i}(\vec{a}_0, \dots, \vec{a}_{n-1}),\ s^{\prod \boldsymbol{A}_i}(\vec{a}_0, \dots, \vec{a}_{n-1})\big)\\
&\qquad = \big\{\, i \in I : t^{\boldsymbol{A}_i}\big(\vec{a}_0(i), \cdots, \vec{a}_{n-1}(i)\big) = s^{\boldsymbol{A}_i}\big(\vec{a}_0(i), \cdots, \vec{a}_{n-1}(i)\big) \,\big\}\\
&\qquad \subseteq J.
\end{aligned}$$

So $\mathrm{EQ}\big(t^{\prod \boldsymbol{A}_i}(\vec{a}_0, \dots, \vec{a}_{n-1}),\ s^{\prod \boldsymbol{A}_i}(\vec{a}_0, \dots, \vec{a}_{n-1})\big) \notin \mathcal{F}$ since $J \notin \mathcal{F}$. Hence

$$t^{\left(\prod A_i\right)/\mathit{\Phi}(\mathcal{F})}\big(\vec{a}_0/\mathit{\Phi}(\mathcal{F}), \dots, \vec{a}_{n-1}/\mathit{\Phi}(\mathcal{F})\big) \neq s^{\left(\prod A_i\right)/\mathit{\Phi}(\mathcal{F})}\big(\vec{a}_0/\mathit{\Phi}(\mathcal{F}), \dots, \vec{a}_{n-1}/\mathit{\Phi}(\mathcal{F})\big),$$

and hence $\big(\prod_{i\in I} \boldsymbol{A}_i\big)/\mathit{\Phi}(\mathcal{F}) \nvDash \varepsilon$. □