3.3. Reduced Products and Ultraproducts. Let $I$ be a nonempty set. Let
\[ \mathcal{P}(I) = \langle \mathcal{P}(I), \cup, \cap, -, \emptyset, I \rangle, \]
where, for every $X \subseteq I$, $\overline{X} = I \setminus X$ is the complement of $X$ relative to $I$. $\mathcal{P}(I)$ is the Boolean algebra of all subsets of $I$. $\mathcal{F} \subseteq \mathcal{P}(I)$ is a filter on or over $I$ if $\mathcal{F}$ is a dual ideal of the lattice $\langle \mathcal{P}(I), \cup, \cap, -, \emptyset, I \rangle$, i.e.,
\begin{enumerate}
\item $\mathcal{F}$ is nonempty;
\item $\mathcal{F}$ is an upper segment, i.e., $X \in \mathcal{F}$ and $X \subseteq Y$ implies $Y \in \mathcal{F}$;
\item $\mathcal{F}$ is closed under intersection, i.e., $X, Y \in \mathcal{F}$ implies $X \cap Y \in \mathcal{F}$.
\end{enumerate}
The set of filters of $I$ is an algebraic closed-set system, since the set of ideals of any lattice forms one. Because of (ii), the condition (i) is equivalent to $I \in \mathcal{F}$. A filter $\mathcal{F}$ is proper if $\mathcal{F} \neq \mathcal{P}(I)$. Because of (ii), $\mathcal{F}$ is proper iff $\emptyset \notin \mathcal{F}$. Thus the union of any chain of proper filters is a proper filter, and consequently Zorn’s lemma can be applied to show that every proper filter $\mathcal{F}$ is included in a maximal proper filter, that is, a proper filter $\mathcal{U}$ such that there is no filter $\mathcal{G}$ such that $\mathcal{U} \subset \mathcal{G} \subset \mathcal{P}(I)$. Maximal proper filters are called ultrafilters.

Examples:
\begin{enumerate}
\item For $J \subseteq I$, $\mathcal{P}(I)[J] = \{ X : J \subseteq X \subseteq I \}$ is the principal filter generated by $J$; for simplicity we normally write $[J]$ for $\mathcal{P}(I)[J]$. A filter $\mathcal{F}$ is principal iff $\mathcal{F} = \mathcal{P}(I)[J]$ in which case $\mathcal{F} = \bigcap \mathcal{F}$. Thus, if $I$ is finite, every filter $\mathcal{F}$ on $I$ is principal. The smallest filter is $[\emptyset] = \{ I \}$ and the largest filter, the improper filter, is $[\emptyset] = \mathcal{P}(I)$.
\item Every nonprincipal filter must be over an infinite set. A subset $X$ of $I$ is cofinite if $\overline{X}$ is finite. Let $\mathcal{C}f$ be the set of all cofinite subsets of $I$. Clearly $I$ is cofinite, and any superset of a cofinite set is cofinite. If $X$ and $Y$ are cofinite, then $X \cap Y = X \cup Y$ is finite, and hence $\mathcal{C}f$ is closed under intersection. Thus, $\mathcal{C}f$ is a filter, $\emptyset$ is cofinite iff $I$ is finite. So $\mathcal{C}f$ is proper if $I$ is infinite. For each $i \in I$, $\{i\}$ is obviously cofinite. Thus $\bigcap \mathcal{C}f \subseteq \bigcap_{i \in I} \{i\} = \emptyset$. Hence $\mathcal{C}f$ is nonprincipal if $I$ is infinite.
\end{enumerate}

Lemma 3.31. Let $I$ be a set, and let $\mathcal{K}$ be an arbitrary set of subsets of $I$. Let $\mathcal{F}$ be the filter generated by $\mathcal{K}$, i.e., $\mathcal{F} := \bigcap \{ \mathcal{G} : \mathcal{G}$ a filter such that $\mathcal{K} \subseteq \mathcal{G} \}$. Then
\[ \mathcal{F} = \{ X : \exists n \in \omega \exists K_1, \ldots, K_n \in \mathcal{K}(K_1 \cap \cdots \cap K_n \subseteq X) \}. \]

Proof. Let $\mathcal{H} = \{ X : \exists n \in \omega \exists K_1, \ldots, K_n \in \mathcal{K}(K_1 \cap \cdots \cap K_n \subseteq X) \}$. If $\mathcal{K}$ is empty, then the only sequence $K_1, \ldots, K_n$ of members of $\mathcal{K}$ is the empty sequence ($n=0$). Then, by definition of the intersection of an empty sequence, $K_1 \cap \cdots \cap K_n = I$. Thus $I \in \mathcal{H}$, and in fact $\mathcal{H} = [I] = \{ I \}$, the smallest filter. And $\mathcal{F} = \{ I \}$, being in this case the intersection of all filters.

Now suppose $\mathcal{K}$ is nonempty. We first verify that $\mathcal{H}$ is a filter that includes $\mathcal{K}$. For each $K_1 \in \mathcal{K}$, $K_1 \subseteq K_1$, and hence $K_1 \in \mathcal{H}$, Thus $\mathcal{K} \subseteq \mathcal{H}$. Suppose $X \in \mathcal{H}$; say $K_1 \cap \cdots \cap K_n \subseteq X$ with $K_1, \ldots, K_n \in \mathcal{K}$. Then $K_1 \cap \cdots \cap K_n \subseteq Y$, and hence $Y \in \mathcal{H}$, for every $Y$ such that $X \subseteq Y$. So $\mathcal{H}$ is an upper segment.

Then $K_1 \cap \cdots \cap K_n \subseteq X$ and $L_1 \cap \cdots \cap L_m \subseteq Y$ with $K_1, \ldots, K_n, L_1, \ldots, L_m \in \mathcal{K}$. Then $K_1 \cap \cdots \cap K_n \cap L_1 \cap \cdots \cap L_m \subseteq X \cap Y$. So $X \cap Y \in \mathcal{H}$, and hence $\mathcal{H}$ is closed under intersection. Thus $\mathcal{H}$ is a filter.
We have seen that \( \mathcal{K} \subseteq \mathcal{H} \). Let \( \mathcal{G} \) be a filter such that \( \mathcal{K} \subseteq \mathcal{G} \). Then \( K_1 \cap \cdots \cap K_n \in \mathcal{G} \) for all \( K_1, \ldots, K_n \in \mathcal{K} \), and hence \( X \in \mathcal{G} \) for every \( X \) such that \( K_1 \cap \cdots \cap K_n \subseteq X \), since \( \mathcal{G} \) is an upper segment. So \( \mathcal{H} \subseteq \mathcal{G} \). Thus \( \mathcal{H} = \mathcal{F} \).

**Corollary 3.32.** Let \( \mathcal{F} \) be a filter over \( I \), and let \( X \in \mathcal{P}(I) \). Let \( \mathcal{G} \) be the smallest filter including \( \mathcal{F} \) that contains \( X \), i.e., the filter generated by \( \mathcal{K} = \mathcal{F} \cup \{X\} \). Then

\[
\mathcal{G} = \{ Y \subseteq I : \exists F \in \mathcal{F} (F \cap X \subseteq Y) \}.
\]

**Proof.** Let \( \mathcal{H} = \{ Y \subseteq I : \exists F \in \mathcal{F} (F \cap X \subseteq Y) \} \). By the lemma, \( \mathcal{G} = \{ Y \subseteq I : \exists n \in \omega \exists K_1, \ldots, K_n \in \mathcal{F} \cup \{X\} (K_1 \cap \cdots \cap K_n \subseteq Y) \} \). Clearly \( \mathcal{H} \subseteq \mathcal{G} \). Let \( Y \in \mathcal{G} \). Then

\[
(26) \quad K_1 \cap \cdots \cap K_n \subseteq Y,
\]

for some \( K_1, \ldots, K_n \in \mathcal{F} \cup \{X\} \). Suppose \( X = K_i \) for some \( i \leq n \); without loss of generality assume \( X = K_n \). Then

\[
K_1 \cap \cdots \cap K_n = \underbrace{K_1 \cap \cdots \cap K_{n-1}}_{F \in \mathcal{F}} \cap X \subseteq Y.
\]

So \( Y \in \mathcal{H} \). If \( X \neq K_i \) for all \( i \leq n \), then \( K_1 \cap \cdots \cap K_n = F \in \mathcal{F} \), and hence (26) implies \( F \cap X \subseteq Y \). So again \( Y \in \mathcal{H} \). So \( \mathcal{G} \subseteq \mathcal{H} \). \( \square \)

**Corollary 3.33.** Let \( \mathcal{K} \subseteq \mathcal{P}(I) \). Then \( \mathcal{K} \) is included in a proper filter and hence an ultrafilter iff, for all \( n \in \omega \) and all \( K_1, \ldots, K_n \in \mathcal{K} \), \( K_1 \cap \cdots \cap K_n \neq \emptyset \).

**Proof.** Exercise. \( \square \)

A set \( \mathcal{K} \) of subsets of a nonempty set \( I \) is said to have the **finite intersection property** if the intersection of every finite subset of \( \mathcal{K} \) is nonempty. By the above corollary, every set of subsets of \( I \) with this property is included in a proper filter.

The following gives a convenient characterization of ultrafilters.

**Theorem 3.34.** Let \( \mathcal{F} \) be a filter over a set \( I \). \( \mathcal{F} \) is an ultrafilter iff

\[
(27) \quad \text{for every } X \subseteq I, \text{ either } X \in \mathcal{F} \text{ or } \overline{X} \in \mathcal{F}, \text{ but not both.}
\]

**Proof.** \( \iff \) Assume (27) holds. Then \( \emptyset \notin \mathcal{F} \) since \( I \in \mathcal{F} \) and \( \emptyset = \overline{I} \). So \( \mathcal{F} \) is proper. Let \( \mathcal{G} \) be a filter such that \( \mathcal{F} \subseteq \mathcal{G} \). Let \( X \in \mathcal{G} \setminus \mathcal{F} \). Then by (27) \( \overline{X} \in \mathcal{F} \subseteq \mathcal{G} \). Thus \( \emptyset = X \cap \overline{X} \in \mathcal{G} \), i.e., \( \mathcal{G} = \mathcal{P}(I) \). Thus \( \mathcal{F} \) is an ultrafilter.

\( \implies \) Suppose \( \mathcal{F} \) is an ultrafilter and \( X \notin \mathcal{F} \). Since \( \mathcal{F} \) is maximal and proper, \( \mathcal{P}(I) \) is smallest filter including \( \mathcal{F} \) that contains \( X \). By Cor. 3.32 \( \mathcal{P}(I) = \{ Y \subseteq I : \exists F \in \mathcal{F} (F \cap X \subseteq Y) \} \). Thus there is an \( F \in \mathcal{F} \) such that \( F \cap X = \emptyset \). So \( F \subseteq \overline{X} \), and hence \( \overline{X} \in \mathcal{F} \). \( \square \)

**Exercises:**

1. A principal filter \([X]\) is an ultrafilter iff \(|X| = 1\).

   The filter \(Cf\) of cofinite sets is never an ultrafilter. If \( I \) is finite, \(Cf\) is the improper filter. If \( I \) is infinite, then \( I \) includes a set \( X \) such that neither \( X \) nor \( \overline{X} \) is finite, and hence neither \( X \) nor \( \overline{X} \) is cofinite.

2. Let \( I \) be infinite. Then \(Cf\) is the smallest nonprincipal filter on \( I \), i.e., for any filter \( \mathcal{F} \) on \( I \), \( \mathcal{F} \) is nonprincipal iff \(Cf \subseteq \mathcal{F} \).
Let $\langle A_i : i \in I \rangle$ be a system of $\Sigma$-algebras, and let $\mathcal{F}$ be a filter on $I$. Define $\Phi(\mathcal{F}) \subseteq (\prod_{i \in I} A_i)^2$ by the condition that

$$\langle (a_i : i \in I), (b_i : i \in I) \rangle \in \Phi(\mathcal{F}) \quad \text{iff} \quad \{ i \in I : a_i = b_i \} \in \mathcal{F},$$

where $EQ(\vec{a}, \vec{b}) := \{ i \in I : a_i = b_i \}$ is called the equality set of $\vec{a}$ and $\vec{b}$. Note that $\langle \vec{a}, \vec{b} \rangle \in \Phi(\mathcal{F})$ iff $EQ(\vec{a}, \vec{b})$ is cofinite, i.e., iff $\{ i \in I : a_i \neq b_i \}$ is finite. It is traditional to say that $\vec{a}$ and $\vec{b}$ are equal “almost everywhere” in this case.

**Lemma 3.35.** $\Phi(\mathcal{F}) \subseteq \text{Co}(\prod_{i \in I} A_i)$ for every filter $\mathcal{F}$ on $I$.

**Proof.** $EQ(\vec{a}, \vec{a}) = I \in \mathcal{F}$. So $\Phi(\mathcal{F})$ is reflexive, and it is symmetric because $EQ(\vec{a}, \vec{b}) = EQ(\vec{b}, \vec{a})$.

I.e., $EQ(\vec{a}, \vec{b}) \cap EQ(\vec{b}, \vec{c}) \subseteq EQ(\vec{a}, \vec{c})$. So if $EQ(\vec{a}, \vec{b})$ and $EQ(\vec{b}, \vec{c})$ are both in $\mathcal{F}$, then so is $EQ(\vec{a}, \vec{c})$. This means that $\Phi(\mathcal{F})$ is transitive.

Let $\sigma \in \Sigma_n$ and $\vec{a}_1, \ldots, \vec{a}_n, \vec{b}_1, \ldots, \vec{b}_n \in \prod_{i \in I} A_i$ such that $EQ(\vec{a}_1, \vec{b}_1), \ldots, EQ(\vec{a}_n, \vec{b}_n) \in \mathcal{F}$. Then as in the proof of transitivity it can be shown that $EQ(\vec{a}_1, \vec{b}_1) \cap \cdots \cap EQ(\vec{a}_n, \vec{b}_n) \subseteq EQ(\sigma \prod A_i, (\vec{a}_1, \ldots, \vec{a}_n), \sigma \prod A_i, (\vec{b}_1, \ldots, \vec{b}_n)) \in \mathcal{F}$. So $\Phi(\mathcal{F})$ has the substitution property.

$\Phi(\mathcal{F})$ is called the filter congruence defined by $\mathcal{F}$.

**Definition 3.36.** Let $\vec{A} = (A_i : i \in I)$ be a system of $\Sigma$-algebras. A $\Sigma$-algebra $B$ is a reduced product of $\vec{A}$ if $B = (\prod_{i \in I} A_i)/\Phi(\mathcal{F})$ for some filter $\mathcal{F}$ on $I$. $B$ is called an ultraproduct of $\vec{A}$ if $\mathcal{F}$ is an ultrafilter.

Note that $B \preceq_r \prod_{i \in I} A_i$, i.e., $B$ is a homomorphic image of $\prod_{i \in I} A_i$, but it is a very special kind of homomorphic image as we shall see. For any $\Sigma$-algebra $C$, we write $C \preceq_r \prod_{i \in I} A_i$ if $C$ is isomorphic to a reduced product of $\vec{A}$; by the First Isomorphism Theorem, $C \preceq_r \prod_{i \in I} A_i$ iff $C$ is a homomorphic image of $\prod_{i \in I} A_i$ by a homomorphism whose relation kernel is a filter congruence. We write $C \preceq_u \prod_{i \in I} A_i$ if $C$ is isomorphic to an ultraproduct of $\vec{A}$.

For any class $K$ of $\Sigma$-algebras,

$$P_r(K) := \{ B : \exists I \exists \vec{A} \in K \bigg( B \preceq_r \prod_{i \in I} A_i \bigg) \}.$$ 

$P_u(K)$ is similarly defined with “$\preceq_u$” in place of “$\preceq_r$”.

Let $I$ be a set and $\mathcal{F}$ a filter on $I$. Let $J \subseteq I$ and define

$$\mathcal{F}|J := \{ F \cap J : F \in \mathcal{F} \}.$$ 

$\mathcal{F}|J$ is a filter on $J$: we verify the three defining properties of a filter.

$J = I \cap J \in \mathcal{F}|J$. Suppose $X \in \mathcal{F}|J$ and $X \subseteq Y \subseteq J$. Let $F \in \mathcal{F}$ such that $X = F \cap J$. Then $F \cup Y \in \mathcal{F}$ and $Y = X \cup Y = (F \cap J) \cup (Y \cap J) = (F \cup Y) \cap J \in \mathcal{F}|J$. Finally, suppose $X, Y \in \mathcal{F}|J$, and let $F, G \in \mathcal{F}$ such that $X = F \cap J$ and $Y = G \cap J$. Then $X \cap Y = (F \cap G) \cap J \in \mathcal{F}|J$.  

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It is easy to see that, if \( J \in \mathcal{F} \), then
\[
\mathcal{F}\mid J = \mathcal{P}(J) \cap \mathcal{F}(= \{ X \subseteq J : X \in \mathcal{F} \}).
\]
The inclusion from right to left holds for all \( J \subseteq I \), without the assumption that \( J \in \mathcal{F} \). Assume, \( X \in \mathcal{F}\mid J \), i.e., \( X = F \cap J \) for some \( F \in \mathcal{F} \). Then \( X \in \mathcal{F} \) since \( J \in \mathcal{F} \).

The following will prove useful in the sequel.

(28) If \( J \in \mathcal{F} \), then \( \forall X \subseteq I (X \in \mathcal{F} \) iff \( X \cap J \in \mathcal{F}\mid J \).

\( X \in \mathcal{F} \) implies \( X \cap J \in \mathcal{F}\mid J \) by the definition of \( \mathcal{F}\mid J \). For the implication in the other direction, assume \( X \cap J \in \mathcal{F}\mid J \). Then, since \( J \in \mathcal{F} \), \( X \cap J \in \mathcal{F} \) by the above characterization of \( \mathcal{F}\mid J \) when \( J \in \mathcal{F} \). Hence \( X \in \mathcal{F} \) since \( \mathcal{F} \) is an upper segment.

**Lemma 3.37.** Let \( \langle A_i : i \in I \rangle \) be a system of \( \Sigma \)-algebras and \( \mathcal{F} \) a filter on \( I \). Then, for each \( J \in \mathcal{F} \),
\[
\left( \prod_{i \in I} A_i \right) / \Phi(\mathcal{F}) \cong \left( \prod_{j \in J} A_j \right) / \Phi(\mathcal{F}\mid J).
\]

**Proof.** Consider the epimorphisms
\[
\prod_{i \in I} A_i \xrightarrow{\pi_j} \prod_{j \in J} A_j \xrightarrow{\Delta_{\mathcal{F}\mid J}} \left( \prod_{j \in J} A_j \right) / \Phi(\mathcal{F}\mid J),
\]
where \( \pi_j(\langle a_i : i \in I \rangle) = \langle a_j : j \in J \rangle \). \( \pi_j \) is the \( J \)-projection function, and it is easily checked that it is an epimorphism; it generalizes the ordinary projection function \( \pi_i \), which can be identified with \( \pi_{\{i\}} \). \( \Delta_{\mathcal{F}\mid J} \) is of course the natural map.

Let \( h = \Delta_{\mathcal{F}\mid J} \circ \pi_j : \prod_{i \in I} A_i \to \left( \prod_{j \in J} A_j \right) / \Phi(\mathcal{F}\mid J) \). Let \( \vec{a} = \langle a_i : i \in I \rangle \) and \( \vec{b} = \langle b_i : i \in I \rangle \).

\[
(\vec{a}, \vec{b}) \in \text{rker}(h) \iff \Delta_{\mathcal{F}\mid J}(\vec{a})|J = \Delta_{\mathcal{F}\mid J}(\vec{b})|J
\]
\[
\iff \langle \vec{a}|J, \vec{b}|J \rangle \in \Phi(\mathcal{F}|J)
\]
\[
\iff \text{EQ}(\vec{a}|J, \vec{b}|J) = \{ j \in J : a_j = b_j \} \in \mathcal{F}|J
\]
\[
\iff \text{EQ}(\vec{a}, \vec{b}) = \{ i \in I : a_i = b_i \} \in \mathcal{F};
\]

this last equivalence holds by (28) since \( \text{EQ}(\vec{a}|J, \vec{b}|J) = \text{EQ}(\vec{a}, \vec{b}) \cap J \)

\[
\iff (\vec{a}, \vec{b}) \in \Phi(\mathcal{F}).
\]
So \( \text{rker}(h) = \Phi(\mathcal{F}) \). Now apply the First Isomorphism Theorem. \( \square \)

By the next lemma, a product \( \prod_{i \in I} A_i \) that is reduced by the filter congruence defined by the filter of cofinite sets is a model of a given identity iff the factor \( A_i \) is a model of the identity for “almost all” \( i \).

**Lemma 3.38.** Let \( \langle A_i : i \in I \rangle \) be a system of \( \Sigma \)-algebras, and let \( \mathcal{F} \) be a filter on \( I \). Let \( \varepsilon \) be an arbitrary \( \Sigma \)-equation. Then
\[
\left( \prod_{i \in I} A_i \right) / \Phi(\mathcal{F}) \vDash \varepsilon \iff \{ i \in I : A_i \vDash \varepsilon \} \in \mathcal{F}.
\]
Proof. Let \( J = \{ i \in I : A_i \models \varepsilon \} \).

\[ \iff \]
Assume \( J \in \mathcal{F} \). Then by Lem. 3.37, \( (\prod_{i \in I} A_i) / \Phi(\mathcal{F}) \cong (\prod_{j \in J} A_j) / \Phi(\mathcal{F}|J) \in \mathbb{H}P(\{ A_j : j \in J \}) \subseteq \mathbb{H}P(\text{Mod}(\varepsilon)) = \text{Mod}(\varepsilon) \); the last equality holds by Thm. 3.20. Thus \((\prod_{i \in I} A_i) / \Phi(\mathcal{F}) \models \varepsilon\).

\[ \implies \]
Suppose \( J \notin \mathcal{F} \). Let \( \varepsilon = t(x_0, \ldots, x_{n-1}) \approx s(x_0, \ldots, x_{n-1}) \). For each \( i \in I \setminus J \), choose \( a_0(i), \ldots, a_{n-1}(i) \in A_i \) such that \( t^{A_i}(a_0(i), \ldots, a_{n-1}(i)) \neq s^{A_i}(a_0(i), \ldots, a_{n-1}(i)) \). This is possible since \( A_i \not\models \varepsilon \). For each \( i \in J \), let \( a_0(i), \ldots, a_{n-1}(i) \) be arbitrary elements of \( A_i \). Let \( \vec{a}_0 = \langle a_0(i) : i \in I \rangle, \ldots, \vec{a}_{n-1} = \langle a_{n-1}(i) : i \in I \rangle \). Recall, that

\[
t\prod_{A_i} (\vec{a}_0, \ldots, \vec{a}_{n-1}) = \langle t^{A_i}(\vec{a}_0(i), \ldots, \vec{a}_{n-1}(i)) : i \in I \rangle \quad \text{and} \quad 
s\prod_{A_i} (\vec{a}_0, \ldots, \vec{a}_{n-1}) = \langle s^{A_i}(\vec{a}_0(i), \ldots, \vec{a}_{n-1}(i)) : i \in I \rangle.
\]

Thus

\[
\text{EQ}\left(t\prod_{A_i} (\vec{a}_0, \ldots, \vec{a}_{n-1}), s\prod_{A_i} (\vec{a}_0, \ldots, \vec{a}_{n-1})\right) = \{ i \in I : t^{A_i}(\vec{a}_0(i), \ldots, \vec{a}_{n-1}(i)) = s^{A_i}(\vec{a}_0(i), \ldots, \vec{a}_{n-1}(i)) \} 
\]

\( \subseteq J \).

So \( \text{EQ}\left(t\prod_{A_i} (\vec{a}_0, \ldots, \vec{a}_{n-1}), s\prod_{A_i} (\vec{a}_0, \ldots, \vec{a}_{n-1})\right) \notin \mathcal{F} \) since \( J \notin \mathcal{F} \). Hence

\[
t\left(\prod_{A_i}/\Phi(\mathcal{F})\right)(\vec{a}_0/\Phi(\mathcal{F}), \ldots, \vec{a}_{n-1}/\Phi(\mathcal{F})) \neq s\left(\prod_{A_i}/\Phi(\mathcal{F})\right)(\vec{a}_0/\Phi(\mathcal{F}), \ldots, \vec{a}_{n-1}/\Phi(\mathcal{F}))
\]

and hence \((\prod_{i \in I} A_i)/\Phi(\mathcal{F}) \not\models \varepsilon\). \( \square \)