2.8. **Direct products.** Of the three basic ways of constructing new Σ-algebras from old ones, the direct product is the only one that increases complexity, or at least the size of the algebras. It is also distinct in that it is a way of combining system of many algebras into a single one.

Let I be a set (possibly empty) and let $\langle\, \boldsymbol{A}_i : i \in I \,\rangle$ be an I-indexed system of nonempty sets. We recall that the *direct* or *Cartesian product* of the system is

$$\prod_{i\in I} A_i = \big\{\, \vec{a} : \vec{a}\colon I \to \bigcup_{i\in I} A_i, \forall\, i \in I\,(\vec{a}(i) \in A_i)\,\big\}.$$

Intuitively, $\prod_{i\in I} A_i$ is the set of all "I-dimensional vectors", that is I-indexed systems of elements, such that the i-component is a member of A_i for each $i \in I$. We will often write a_i for the i-th component of $\vec{a}$, i.e., $a_i = \vec{a}(i)$, so that $\vec{a} = \langle\, a_i : i \in I \,\rangle$,

Definition 2.42. Let be a set (possibly empty) and let $\langle\, \boldsymbol{A}_i : i \in I \,\rangle$ be an I-indexed system of Σ-algebras. By the *direct* or *Cartesian product* of $\langle\, \boldsymbol{A}_i : i \in I \,\rangle$ we mean the Σ-algebra

$$\prod_{i\in I} \boldsymbol{A}_i = \langle \prod_{i\in I} A_i,\ \sigma^{\prod_i A_i} \rangle_{\sigma\in\Sigma},$$

where $\sigma^{\prod_i A_i}(\vec{a}_1, \dots, \vec{a}_n) = \big\langle\, \sigma^{\boldsymbol{A}_i}\big(\vec{a}_1(i), \dots, \vec{a}_n(i)\big) : i \in I \,\big\rangle$ for each $\sigma \in \Sigma_n$ and all $\vec{a}_1, \dots, \vec{a}_n \in \prod_{i\in I} A_i$. The algebras $\boldsymbol{A}_i$ are called (*direct*) *factors* of $\prod_{i\in I} \boldsymbol{A}_i$.

By the *I-th direct*, or *I-th Cartesian, product* of a Σ-algebra $\boldsymbol{A}$ we mean $\boldsymbol{A}^{\boldsymbol{I}} = \prod_{i\in I} \boldsymbol{A}_i$ were $\boldsymbol{A}_i = \boldsymbol{A}$ for every $i \in I$.

Remarks:

(1) We normally write $\boldsymbol{A}_1 \times \cdots \times \boldsymbol{A}_n$ for $\prod_{i\in\{1,2,\dots,n\}} \boldsymbol{A}_i$.

(2) If the index set I is empty, then

$$\prod_{i\in I} A_i = \big\{\, \vec{a} : \vec{a}\colon I \to \bigcup_{i\in I}, \forall\, i \in I\,(\vec{a}(i) \in A_i)\,\big\} = \{\, \vec{a} : \vec{a}\colon \emptyset \to \emptyset \,\} = \{\emptyset\},$$

where $\emptyset$ is the empty function, the function with empty domain. Thus $\prod_{i\in\emptyset} \boldsymbol{A}_i$ is a one-element Σ-algebra. All one-element Σ-algebras are isomorphic; they are called *trivial Σ-algebras.* They all have the form $\langle\, \{a\}, \sigma^{\boldsymbol{A}} \rangle_{\sigma\in\Sigma}$, where $\sigma^{\boldsymbol{A}}(a, \cdots, a) = a$ for all $\sigma \in \Sigma$.

Let $\boldsymbol{A} = \langle A, \vee^{\boldsymbol{A}}, \wedge^{\boldsymbol{A}}, 0^{\boldsymbol{A}}, 1^{\boldsymbol{A}} \rangle$ and $\boldsymbol{B} = \langle B, \vee^{\boldsymbol{B}}, \wedge^{\boldsymbol{B}}, 0^{\boldsymbol{B}}, 1^{\boldsymbol{B}} \rangle$ be bounded lattices.

$$\boldsymbol{A} \times \boldsymbol{B} = \langle A \times B, \vee^{\boldsymbol{A}\times\boldsymbol{B}}, \wedge^{\boldsymbol{A}\times\boldsymbol{B}}, 0^{\boldsymbol{A}\times\boldsymbol{B}}, 1^{\boldsymbol{A}\times\boldsymbol{B}} \rangle,$$

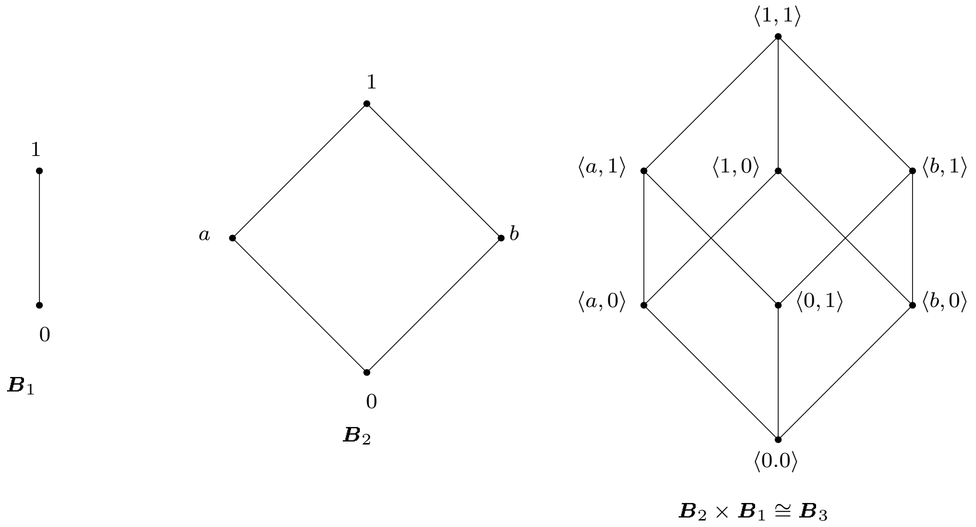
where $\langle a, b \rangle \vee^{\boldsymbol{A}\times\boldsymbol{B}} \langle a', b' \rangle = \langle a \vee^{\boldsymbol{A}} a',\, b \vee^{\boldsymbol{B}} b' \rangle$ and $\langle a, b \rangle \wedge^{\boldsymbol{A}\times\boldsymbol{B}} \langle a', b' \rangle = \langle a \wedge^{\boldsymbol{A}} a',\, b \wedge^{\boldsymbol{B}} b' \rangle$ and $0^{\boldsymbol{A}\times\boldsymbol{B}} = \langle 0^{\boldsymbol{A}},\, 0^{\boldsymbol{B}} \rangle$ and $1^{\boldsymbol{A}\times\boldsymbol{B}} = \langle 1^{\boldsymbol{A}},\, 1^{\boldsymbol{B}} \rangle$.

We note that

$$\langle a, b \rangle \leq^{\boldsymbol{A}\times\boldsymbol{B}} \langle a', b' \rangle \quad \text{iff} \quad a \leq^{\boldsymbol{A}} a' \text{ and } b \leq^{\boldsymbol{B}} b'.$$

This follows from the following simple computation. $\langle a, b \rangle \leq^{\boldsymbol{A}\times\boldsymbol{B}} \langle a', b' \rangle$ iff $\langle a, b \rangle \wedge^{\boldsymbol{A}\times\boldsymbol{B}} \langle a', b' \rangle = \langle a, b \rangle$ iff $\langle a \wedge^{\boldsymbol{A}} a',\, b \wedge^{\boldsymbol{B}} b' \rangle = \langle a', b' \rangle$ iff $a \wedge^{\boldsymbol{A}} a' = a$ and $b \wedge^{\boldsymbol{B}} b' = b$ iff $a \leq^{\boldsymbol{A}} a'$ and $b \leq^{\boldsymbol{B}} b'$. See Figure 17.

In general, for any system $\langle\, \boldsymbol{L}_i : i \in I \,\rangle$ of lattices, $\vec{a} \leq \vec{b}$ iff, for all $i \in I$, $\vec{a}(i) \leq \vec{b}(i)$ (exercise).



$\boldsymbol{B}_1$

$\boldsymbol{B}_2$

$\boldsymbol{B}_2 \times \boldsymbol{B}_1 \cong \boldsymbol{B}_3$

FIGURE 17

Theorem 2.43 (Generalized Communitive Law)**.** *Let* $\langle\, \boldsymbol{A}_i : i \in I \,\rangle$ *be a system of* Σ*-algebras and let* $h\colon I \rightarrowtail\!\!\!\!\twoheadrightarrow I$ *be a bijection between* I *and itself, i.e., a* permutation*. Then* $\prod_{i\in I} \boldsymbol{A}_i \cong \prod_{i\in I} \boldsymbol{A}_{h(i)}$*.*

Proof. The map $\vec{a} \mapsto \langle\, \vec{a}\big(h(i)\big) : i \in I \,\rangle$ is an isomorphism. This is left as an exercise. ☐

Corollary 2.44. (i) $\boldsymbol{A} \times \boldsymbol{B} \cong \boldsymbol{B} \times \boldsymbol{A}$.
(ii) $\boldsymbol{A} \times \boldsymbol{B} \times \boldsymbol{C} \cong \boldsymbol{A} \times \boldsymbol{C} \times \boldsymbol{B} \cong \boldsymbol{B} \times \boldsymbol{A} \times \boldsymbol{C} \cong \boldsymbol{B} \times \boldsymbol{C} \times \boldsymbol{A} \cong \boldsymbol{C} \times \boldsymbol{A} \times \boldsymbol{B} \cong \boldsymbol{C} \times \boldsymbol{B} \times \boldsymbol{A}$.

Proof. For example: Let $\boldsymbol{D}_1 = \boldsymbol{A}$, $\boldsymbol{D}_2 = \boldsymbol{B}$, and $\boldsymbol{D}_3 = \boldsymbol{C}$, and let $h(1) = 2$, $h(2) = 3$, and $h(3) = 1$. Then $\boldsymbol{A} \times \boldsymbol{B} \times \boldsymbol{C} = \prod_{i\in\{1,2,3\}} \boldsymbol{D}_i$ and $\boldsymbol{B} \times \boldsymbol{C} \times \boldsymbol{A} = \prod_{i\in\{1,2,3\}} \boldsymbol{A}_{h(i)}$. ☐

Theorem 2.45 (Generalized Associative Law)**.** *Let* $\langle\, \boldsymbol{A}_i : i \in I \,\rangle$*, and let* $\{\, I_j : j \in J \,\}$ *be a partition of* I *then*

$$\prod_{j\in J}\Big(\prod_{i\in I_j} \boldsymbol{A}_i\Big) \cong \prod_{i\in I} \boldsymbol{A}_i.$$

Proof. The map $\vec{a} \mapsto \big\langle\, \langle \vec{a}(i) : i \in I_j \,\rangle : j \in J \,\big\rangle$ is an isomorphism (exercise). ☐

Corollary 2.46. $(\boldsymbol{A} \times \boldsymbol{B}) \times \boldsymbol{C} \cong \boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) \cong \boldsymbol{A} \times \boldsymbol{B} \times \boldsymbol{C}$. ☐

In the sequel, unless explicitly stated otherwise, $\langle\, \boldsymbol{A}_i : i \in I \,\rangle$ will be an I-indexed system of Σ-algebras, and $\vec{a}, \vec{b}, \vec{c}$ will represent arbitrary elements of $\prod_{i\in I} A_i$. $\prod_{i\in I} \boldsymbol{A}_i$ will often be written in the simpler form $\prod_i \boldsymbol{A}_i$ or, even more simply, $\prod \boldsymbol{A}_i$.

Definition 2.47. For each $i \in I$, $\pi\colon \prod_{i\in I} A_i \twoheadrightarrow A_i$ is defined by $\pi_i(\vec{a}) = \vec{a}(i)$ for each $i \in I$. φ_i is called the *i-th projection*.

Special case: $\pi_1\colon A \times B \twoheadrightarrow A$ and $\pi_2\colon A \times B \twoheadrightarrow B$.

Note that $\vec{a} = \vec{b}$ iff, for all $i \in I$, $\pi_i(\vec{a}) = \pi_i(\vec{b})$.

$$\begin{tikzcd}
& \boldsymbol{C} \arrow[ld, "h_j"'] \arrow[d, dashed, "!\bar{h}"'] \arrow[rd, "h_k"] & \\
\boldsymbol{A}_j & \prod\limits_{i\in I} \boldsymbol{A}_i \arrow[l, "\pi_j"'] \arrow[r, "\pi_k"] & \boldsymbol{A}_k
\end{tikzcd}$$

FIGURE 18

$\pi_i\colon \prod_{i\in I}\boldsymbol{A}_i \twoheadrightarrow \boldsymbol{A}_i$ is an epimorphism. We check this. $\pi_i\big(\sigma^{\prod_i \boldsymbol{A}_i}(\vec{a}_1,\dots,\vec{a}_n)\big) = \pi_i\big(\big\langle\, \sigma^{\boldsymbol{A}_j}(\vec{a}_1(j),\dots,\vec{a}_n(j)) : j\in J\,\big\rangle\big) = \sigma^{\boldsymbol{A}_i}\big(\vec{a}_1(i),\dots,\vec{a}_n(i) = \sigma^{\boldsymbol{A}_i}\big(\pi_i(\vec{a}_i),\dots,\pi_i(\vec{a}_n)\big)$.

Theorem 2.48 (Categorical Product Property)**.** *Let $\langle\, \boldsymbol{A}_i : i\in I\,\rangle$ be a system of Σ-algebras. For every Σ-algebra $\boldsymbol{C}$ and every system $\vec{h} = \langle\, h_i : i\in I\,\rangle \in \prod_{i\in I}\operatorname{Hom}(\boldsymbol{C},\boldsymbol{A}_i)$ of homomorphisms of $\boldsymbol{C}$ into the $\boldsymbol{A}_i$, there exists a unique $\bar{h}\in\operatorname{Hom}\big(\boldsymbol{C},\prod_{i\in I}\boldsymbol{A}_i\big)$ such that $h_i = \pi_i\circ\bar{h}$ for every $i\in I$, i.e., such that the diagram in Figure 18 is commutative.*

Proof. Define $\bar{h}$ by $\bar{h}(c) = \langle\, h_i(c) : i\in I\,\rangle$.

$$\begin{aligned}
\bar{h}\big(\sigma^{\boldsymbol{C}}(c_1,\dots,c_n)\big) &= \langle h_i\big(\sigma^{\boldsymbol{C}}(c_1,\dots,c_n)\big) : i\in I\,\rangle \\
&= \big\langle \sigma^{\boldsymbol{A}_i}\big(h_i(c_1),\dots,h_i(c_n)\big) : i\in I\,\big\rangle \\
&= \big\langle\, \sigma^{\boldsymbol{A}_i}\big(\bar{h}(c_1)(i),\dots,\bar{h}(c_n)(i)\big) : i\in I\,\big\rangle \\
&= \sigma^{\prod\boldsymbol{A}_i}\big(\bar{h}(c_1),\dots,\bar{h}(c_n)\big).
\end{aligned}$$

So $\bar{h}$ is a homomorphism.

For every $c\in C$ and every $j\in I$, $(\pi_j\circ\bar{h})(c) = \pi_j\big(\bar{h}(c)\big) = \pi_j\big(\langle\, h_i(c) : i\in I\,\rangle\big) = h_j(c)$. So $\pi_i\circ\bar{h} = h_i$ for each $i\in I$. ☐

Corollary 2.49. *$\boldsymbol{C} \succcurlyeq ; \subseteq \prod_{i\in I}\boldsymbol{A}_i$ iff, for every $i\in I$, $\boldsymbol{C}\succcurlyeq ; \subseteq \boldsymbol{A}_i$.*

Proof.

$$\begin{aligned}
\forall\, i\in I\,(\boldsymbol{C}\succcurlyeq ; \subseteq \boldsymbol{A}_i) &\Longleftrightarrow \forall\, i\in I\,\exists\, h_{i\,i}\colon \boldsymbol{C}\to\boldsymbol{A}_i) \\
&\Longleftrightarrow \exists\,\bar{h}\,(\bar{h}\colon \boldsymbol{C}\to\prod_{i\in I}\boldsymbol{A}_i) \\
&\Longleftrightarrow \boldsymbol{C}\succcurlyeq ; \prod_{i\in I}\boldsymbol{A}_i.
\end{aligned}$$

☐

Corollary 2.50. *Let $j\in I$. Then $\boldsymbol{A}_j\cong ; \subseteq \prod_{i\in I}\boldsymbol{A}_i$ iff, for each $i\in I\setminus\{j\}$, $\boldsymbol{A}_j\succcurlyeq ; \subseteq \boldsymbol{A}_i$.*

Proof. $\Longrightarrow$ Assume $\boldsymbol{A}_j \overset{h}{\cong} \boldsymbol{B}\subseteq\prod_{i\in I}\boldsymbol{A}_i$. Then $\pi_i\circ h\colon \boldsymbol{A}_j\to\boldsymbol{A}_i$ for each $i\in I\setminus\{j\}$.

$\Longleftarrow$ Assume $h_i\colon \boldsymbol{A}_j\to\boldsymbol{A}_i$ for each $i\in I\setminus\{j\}$. Let $h_j = \Delta_{A_j}\colon \boldsymbol{A}_j\cong\boldsymbol{A}_j$. By the Categorical Product Property, there is a $\bar{h}\colon \boldsymbol{A}_j\to\prod_{i\in I}\boldsymbol{A}_i$ such that $\pi_i\circ\bar{h} = h_i$ for each

$i \in I$. For every $a \in A_j$, $\pi_j\big(\bar{h}(a)\big) = (\pi_j \circ \bar{h})(a) = h_j(a) = a$. Thus $\bar{h}$ is injective and hence $\boldsymbol{A}_j \overset{\bar{h}}{\cong} \bar{h}(\boldsymbol{A}_j) \subseteq \prod_{i\in I} \boldsymbol{A}_i$. So $\boldsymbol{A}_j \cong \boldsymbol{;} \subseteq \prod_{i\in I} \boldsymbol{A}_i$. □

In particular, if every $\boldsymbol{A}_i$ has a trivial subalgebra, then, for every $j \in I$, $\boldsymbol{A}_j$ is isomorphic to a subalgebra of $\prod_{i\in I} \boldsymbol{A}_i$. This is so because the function that maps all of A_j to the unique element of the trivial subalgebra of $\boldsymbol{A}_i$ for each $i \in I \setminus \{j\}$ is a homomorphism. For this reason every group is isomorphic to a subgroup of any direct product of groups which includes it as one of its direct factors.

Definition 2.51. Let $\boldsymbol{A}$ be a Σ-algebra. A system $\vec{\alpha} = \langle\, \alpha_i : i \in I \,\rangle$ of congruence relations of $\boldsymbol{A}$ is called a *factor congruence system* (FAC) *for* $\boldsymbol{A}$ if

(i) $\bigcap_{i\in I} \alpha_i = \Delta_A$, and,

(ii) for every $\vec{a} = \langle\, a_i : i \in I \,\rangle \in A^I$, $\bigcap_{i\in I}(a_i/\alpha_i) \neq \emptyset$.

Notice that $b \in \bigcap_{i\in I} a_i/\alpha_i$ iff b is a solution of the system of congruence equations

$$\forall\, i \in I\, (x \equiv a_i \pmod{\alpha_i}). \tag{25}$$

Notice further that condition (i) implies that any solution of this system of congruence equations is unique.

Thus condition (ii) in the above definition is equivalent to the requirement that the system (25) of congruence equations has a solution. For this reason condition (ii) is called the *Chinese Remainder Property*, the CRP for short. If $\langle n_1, \dots, n_k\rangle$ is a finite system of pairwise relatively prime integers, and if for each $i \leq k$, α_i is the congruence on the ring of integers $\mathbb{Z}$ defined by $\langle a, b\rangle \in \alpha_i$ if $a \equiv b (\text{mod } n_i)$, then the classical Chinese Remainder Theorem says that $\langle \alpha_1, \dots, \alpha_k\rangle$ has the CRP.

The following theorem characterizes those Σ-algebras $\boldsymbol{B}$ that are isomorphic to a given direct product in terms of the congruences of $\boldsymbol{B}$.

Theorem 2.52. $\boldsymbol{B} \cong \prod_{i\in I} \boldsymbol{A}_i$ *iff there exists a factor congruence system* $\vec{\alpha} = \langle\, \alpha_i : i \in I \,\rangle$ *such that, for all* $i \in I$, $\boldsymbol{A}_i \cong \boldsymbol{B}/\alpha_i$.

Proof. Suppose that $h\colon \boldsymbol{B} \cong \boldsymbol{C}$ and $\vec{\alpha} = \langle\, a_i : i \in I \,\rangle$ is a FCS for $\boldsymbol{C}$ such that $\boldsymbol{C}/\alpha_i \cong \boldsymbol{A}_i$ for each $i \in I$. Then we claim that $h^{-1}(\vec{\alpha}) := \langle\, h^{-1}(\alpha_i) : i \in I \,\rangle$ is a FCS for $\boldsymbol{B}$ such that $\boldsymbol{B}/h^{-1}(\alpha_i) \cong \boldsymbol{A}_i$ for each $i \in I$. (Recall that $h^{-1}(\alpha_i) = \{\, \langle b, b'\rangle \in B^2 : \langle h(b), h(b')\rangle \in \alpha_i \,\}$.) By set theory, $\bigcap_{i\in I} a_i/\alpha_i = h^{-1}\big(\bigcap_{i\in I} \alpha_i\big) = h^{-1}(\Delta_C) = \Delta_B$ (since h is a bijection). Also, consider any $\langle\, b_i : i \in I \,\rangle \in B^I$, and let $c \in C$ such that $c \equiv h(b_i) (\text{mod } \alpha_i)$ for each $i \in I$, i.e., $hh^{-1}(c) \equiv h(b_i) (\text{mod } \alpha_i)$ for each $i \in I$. Thus $h^{-1}(c)$ is a solution of the system of congruence equations $x \equiv b_i (\text{mod } h^{-1}(\alpha_i))$, $i \in I$. Finally, the mapping from B to A/α_i such that $b \mapsto h(b)/\alpha_i$ is a surjective homomorphism with relation kernel $h^{-1}(\alpha_i)$. Thus $\boldsymbol{B}/h^{-1}(\alpha_i) \cong \boldsymbol{A}_i$ by the First Isomorphism Theorem. This establishes the claim.

We now verify the conclusion of the theorem.

$\Longrightarrow$ Assume $\boldsymbol{B} \cong \prod_{i\in I} \boldsymbol{A}_i$. By the above claim we can assume without loss of generality that $\boldsymbol{B} = \prod_{i\in I} \boldsymbol{A}_i$. Let $\alpha_i = \text{rker}(\pi_i)$ for each $i \in I$. Note that,for each $i \in I$, $\vec{a}\, \alpha_i\, \vec{b}$ iff $\pi_i(\vec{a}) = \pi_i(\vec{b})$ iff $\vec{a}(i) = \vec{b}(i)$. So $\vec{a}\ \bigcap_{i\in I} \alpha\ \vec{b}$ iff $\forall\, i \in I\, (\vec{a}\, \alpha_i\, \vec{b})$ iff $\forall\, i \in I\, \big(\vec{a}(i) = \vec{b}(i)\big)$ iff $\vec{a} = \vec{b}$. So $\bigcap_{i\in I} \vec{a}_i = \Delta_A$.

Consider any $\langle\, \vec{a}_i : i \in I\,\rangle \in \big(\prod_{i\in I} A_i\big)^I$, and let $\vec{b} = \langle\, \vec{a}_i(i) : i \in I\,\rangle$. Then $\vec{a}_i(i) = \vec{b}(i)$ for all $i \in I$. So $\vec{a}_i\, \alpha_i\, \vec{b}$ for all $i \in I$, i.e., $\vec{b} = \bigcap_{i\in I} \vec{a}_i/\alpha_i$. And $\big(\prod_{i\in I} \boldsymbol{A}_i\big)/\alpha_i \cong \boldsymbol{A}_i$ by the First Isomorphism Theorem.

$\Longleftarrow$ Assume α is a factor congruence system for $\boldsymbol{B}$ such that $\boldsymbol{B}/\alpha_i \cong \boldsymbol{A}_i$ for all $i \in I$. Let $h_i \colon \boldsymbol{B} \twoheadrightarrow \boldsymbol{A}_i$ such that $\alpha_i = \operatorname{rker}(h_i)$. By the Categorical Product Property there exists a unique $\bar{h}\colon \boldsymbol{B} \to \prod_{i\in I} \boldsymbol{A}_i$ such that $\pi_i \circ \bar{h} = h_i$ for all $i \in I$, i.e., $\bar{h}(b)(i) = h_i(b)$ for each $b \in B$ and all $i \in I$. Thus, for all $b, b' \in B$,

$$\begin{aligned} \bar{h}(b) = \bar{h}(b') \quad &\text{iff} \quad \forall\, i \in I\,\big(h_i(b) = h_i(b')\big)\\ &\text{iff} \quad \forall\, i \in I\,(b\,\alpha_i\, b')\\ &\text{iff} \quad b \bigcap_{i\in I} \alpha_i\, b'\\ &\text{iff} \quad b = b'. \end{aligned}$$

So $\bar{h}$ is injective. Consider any $\vec{a} = \langle\, a_i : i \in I\,\rangle \in \prod_{i\in I} A_i$. For each $i \in I$ choose $b_i \in B$ such that $h_i(b_i) = a_i$. By the Chinese Remainder Property, there is a $b \in B$ such that, for every $i \in I$, $b \equiv_{\alpha_i} a_i$, i.e., $h_i(b) = a_i$ for every $i \in I$. So $\bar{h}(b) = \langle\, h_i(b) : i \in I\,\rangle = \langle\, a_i : i \in I\,\rangle = \vec{a}$. Thus $\bar{h}$ is surjective. □

Theorem 2.53. *let $\vec{\alpha} = \langle\, \alpha_i : i \in I\,\rangle$ be a factor congruence system for $\boldsymbol{A}$, and let $\{\, I_j : j \in J\,\}$ be a partition of I. Let $\beta_j = \bigcap_{i\in I_j} \alpha_i$, for all $j \in J$. Then $\vec{\beta} = \langle\, \beta_j : j \in J\,\rangle$ is also a factor congruence system for $\boldsymbol{A}$. In particular, for each $j \in I$, let $\hat{\alpha}_j = \bigcap_{i\in I\setminus\{j\}} \alpha_i$. Then $\langle\, \alpha_j, \hat{\alpha}_j\,\rangle$ is a factor congruence system for $\boldsymbol{A}$.*

Proof. $\bigcap_{j\in J} \beta_j = \bigcap_{j\in J} \bigcap_{i\in I_j} \alpha_i = \bigcap_{i\in I} \alpha_i = \Delta_A$. Let $\langle\, a_j : j \in J\,\rangle \in A^J$. By the CRP for $\vec{\alpha}$ there is a $b \in A$ such that $\forall\, j \in J\, \forall\, i \in I_j\, (b \equiv_{\alpha_i} a_j)$. Thus $\forall\, j \in J\, \big(b \equiv_{\bigcap_{i\in I_j} \alpha_i} a_j\big)$. I.e., $b \equiv_{\beta_i} a_j$. So, $\vec{\beta}$ has the Chinese Remainder Property. □

Definition 2.54. $\alpha \in \operatorname{Co}(\boldsymbol{A})$ is a *factor congruence of* $\boldsymbol{A}$ if there exists a factor congruence system $\langle\, \beta_i : i \in I\,\rangle$ with $|I| \geq 2$ such that $\alpha = \beta_i$ for some $i \in I$.

Equivalently, by Thm. 2.53, α is a factor congruence if there is a $\hat{\alpha} \in \operatorname{Co}(\boldsymbol{A})$ such that $\langle \alpha, \hat{\alpha}\rangle$ is a factor congruence. α and $\hat{\alpha}$ are *complementary factor congruences of* $\boldsymbol{A}$.

Theorem 2.55. *Let $\alpha, \hat{\alpha} \in \operatorname{Co}\boldsymbol{A}$. α and $\hat{\alpha}$ are* complementary factor congruences *iff $\alpha \cap \hat{\alpha} = \Delta_A$ and $\alpha \,;\, \hat{\alpha} = \nabla_A$.*

Proof.

$$\begin{aligned} \alpha \,;\, \hat{\alpha} = \nabla_A \quad &\text{iff} \quad \forall\, \langle a_1, a_2\rangle \in A^2\, (a_1\, (\alpha \,;\, \hat{\alpha})\, a_2)\\ &\text{iff} \quad \forall\, \langle a_1, a_2\rangle \in A^2\, \exists\, b \in A\, (b\, \alpha\, a_1 \text{ and } b\, \hat{\alpha}\, a_2)\\ &\text{iff} \quad \langle \alpha, \hat{\alpha}\rangle \text{ has the CRP.} \end{aligned}$$

□

Definition 2.56. A Σ algebra $\boldsymbol{A}$ is *directly indecomposable* or *directly irreducible* (DI) if, for every system of Σ-algebras $\langle\, \boldsymbol{B}_i : i \in I\,\rangle$, $\boldsymbol{A} \cong \prod_{i\in I} \boldsymbol{B}_i$ implies $\boldsymbol{B}_k$ is nontrivial for at exactly one $k \in I$.

If $\boldsymbol{A}$ is directly indecomposable, then $\boldsymbol{A} \cong \prod_{I \in i} \boldsymbol{B}_I$ implies that $\boldsymbol{A}$ is isomorphic to one of the $\boldsymbol{B}_i$, namely the one such that all the other direct factors are trivial. We see in Cor. 2.58 below that this is also a sufficient condition for direct idecomposability if $\boldsymbol{A}$ is finite.

Theorem 2.57. *Let $\boldsymbol{A}$ be a nontrivial Σ-algebra. The following three conditions are equivalent.*

(i) *$\boldsymbol{A}$ is directly indecomposable.*
(ii) *$\boldsymbol{A} \cong \boldsymbol{B} \times \boldsymbol{C}$ implies either $\boldsymbol{B}$ or $\boldsymbol{C}$ is trivial.*
(iii) *$\boldsymbol{A}$ has exactly two factor congruence relations, more precisely, the only two factor congruences of $\boldsymbol{A}$ are Δ_A and ∇_A.*

Proof. (i) $\Longrightarrow$ (ii): trivial

(ii) $\Longrightarrow$ (iii). Let α and $\hat{\alpha}$ be complementary congruences of $\boldsymbol{A}$. Then $\boldsymbol{A} \cong \boldsymbol{A}/\alpha \times \boldsymbol{A}/\hat{\alpha}$. By assumption $\boldsymbol{A}/\alpha$ or $\boldsymbol{A}/\hat{\alpha}$ is trivial. In the first case we have $\alpha = \nabla_A$ and hence $\hat{\alpha} = \nabla_A \cap \hat{\alpha} = \alpha \cap \hat{\alpha} = \Delta_A$. If $\boldsymbol{A}/\hat{\alpha}$ is trivial, then $\hat{\alpha} = \nabla_A$ and $\alpha = \Delta_A$. So Δ_A and ∇_A are the only factor congruences of $\boldsymbol{A}$.

(iii) $\Longrightarrow$ (i). Suppose $\boldsymbol{A} \cong \prod_{i \in I} \boldsymbol{B}_i$. Let $\langle\, \alpha_i : i \in I \,\rangle$ be a factor congruence system such that $\boldsymbol{B}_i \cong \boldsymbol{A}/\alpha_i$ for each $i \in I$. By assumption each α_i is either Δ_A or ∇_A. They all cannot be ∇_A since otherwise each $\boldsymbol{B}_i$ is trivial which is impossible since $\boldsymbol{A}$ is nontrivial. So $\alpha_k = \Delta_A$ for at least one $k \in I$. For each $i \in I$ let $\hat{\alpha}_i = \bigcap_{j \in I \setminus \{i\}} \alpha_j$. Note that, for each $i \in I \setminus \{k\}$, $\hat{\alpha}_i \subseteq \alpha_k = \Delta_A$ and hence $\hat{\alpha}_i = \Delta_A$. But by Thm. 2.53 α_i and $\hat{\alpha}_i$ are complementary congruences. So $\alpha_i = \nabla_A$ and hence $\boldsymbol{B}_i$ is trivial for all $i \in I \setminus \{k\}$. □

Every simple algebra is directly indecomposable. An example of a nonsimple algebra that is directly indecomposable is the Abelian group $\mathbb{Z}_{p^n} = \mathbb{Z}/\equiv (\mathrm{mod}\ p^n)$ for each prime p and positive integer n. The only (normal) subgroups are $p^k\mathbb{Z}_{p^n}$ for $k \leq n$ and hence the only congruence relations are $\equiv (\mathrm{mod}\ p^k)/\equiv (\mathrm{mod}\ p^n)$ for $k \leq n$. So the lattice of congruence relations is linearly ordered and hence the only factor congruences are

$$\Delta_{\mathbb{Z}_{p^n}} = \ \equiv\ (\mathrm{mod}\ p^n)/\equiv\ (\mathrm{mod}\ p^n) \quad \text{and} \quad \nabla_{\mathbb{Z}_{p^n}} = \ \equiv\ (\mathrm{mod}\ p^0)/\equiv\ (\mathrm{mod}\ p^n).$$

The fact that the $\mathbb{Z}_{p^n}$ are directly indecomposable, and in fact the only directly indecomposable finitely generated Abelian groups, can also be obtained from the Fundamental Theorem of Abelian Groups.

Corollary 2.58. *Let $\boldsymbol{A}$ be a finite, nontrivial Σ-algebra. The following three conditions are equivalent.*

(i) *$\boldsymbol{A}$ is directly indecomposable.*
(ii) *For every system of Σ-algebras $\langle\, \boldsymbol{B}_i : i \in I \,\rangle$, $\boldsymbol{A} \cong \prod_{i \in I} \boldsymbol{B}_i$ implies $\boldsymbol{B}_k \cong \boldsymbol{A}$ for some $k \in I$.*
(iii) *$\boldsymbol{A} \cong \boldsymbol{B} \times \boldsymbol{C}$ implies either $\boldsymbol{B} \cong \boldsymbol{A}$ or $\boldsymbol{C} \cong \boldsymbol{A}$.*

Proof. Suppose $\boldsymbol{A} \cong \prod_{i \in I} \boldsymbol{B}_i$. If $\boldsymbol{A} \cong \boldsymbol{B}_k$ for some $k \in I$, then $|B_k| = |A|$ and hence, since $\boldsymbol{A}$ is finite, $|B_i| = 1$ for all $i \in I \setminus \{k\}$. Conversely, if $|B_i| = 1$ for all $i \in I \setminus \{k\}$, then $|B_k| = |A|$, and hence the projection function π_k is an isomorphism between $\boldsymbol{A}$ and $\boldsymbol{B}_k$ by the pigeon-hole principle, because $\boldsymbol{A}$ is finite. So the conditions (i) and (ii) are equivalent.

By essentially the same argument, if $\boldsymbol{A} \cong \boldsymbol{B} \times \boldsymbol{C}$, then $\boldsymbol{B}$ is trivial iff $\boldsymbol{A} \cong \boldsymbol{C}$, and $\boldsymbol{C}$ is trivial iff $\boldsymbol{A} \cong \boldsymbol{B}$. So condition (iii) is equivalent to Thm 2.57(ii). □

Neither of the conditions (ii) or (iii) of the corollary is equivalen to direct indecomposability for arbitrary Σ-algebras. In particular, it follows from the remarks at the end of the chapter that every countably infinite left-trivial semigroup satisfies both (ii) and (iii), but none of these algebras is directly irreducible.

We prove in Thm. 2.61 below that every finite Σ-algebra is a direct product of directly indecomposable algebras. But it is shown in subsequent remarks that this is not the case for all infinite Σ-algebras.

For any class K of Σ-algebras define

$$\mathbf{P}(\mathsf{K}) := \{\, \boldsymbol{B} : \exists \langle\, \boldsymbol{A}_i : i \in I \,\rangle \in \mathsf{K}^I \,\big(\boldsymbol{B} \cong \prod_{i\in I} \boldsymbol{A}_i\big) \,\}.$$

We show that $\mathbf{P}$ is a closure operator on $\mathsf{Alg}(\Sigma)$. For each $\boldsymbol{A} \in \mathsf{K}$, $\boldsymbol{A} \cong \prod_{i\in I} \boldsymbol{A}_i$ where $I = \{0\}$ and $\boldsymbol{A}_0 = \boldsymbol{A}$. So $\mathsf{K} \subseteq \mathbf{P}(\mathsf{K})$. By the generalized associative law

$$\prod_{j\in J}\Big(\prod_{i\in I_j} \boldsymbol{A}_{ij}\Big) = \prod_{\langle i,j\rangle \in \bigcup_{j\in J}(I_j \times \{j\})} \boldsymbol{A}_{ij}.$$

So $\mathbf{P}\,\mathbf{P}(\mathsf{K}) = \mathbf{P}(\mathsf{K})$. Finally, it is obvious that $\mathsf{K} \subseteq \mathsf{L}$ implies $\mathbf{P}(\mathsf{K}) \subseteq \mathbf{P}(\mathsf{L})$. $\mathbf{P}$ is not algebraic (exercise).

Theorem 2.59. *Let* K *be any class of* Σ*-algebras.*

(i) $\mathbf{P}\,\mathbf{H}(\mathsf{K}) \subseteq \mathbf{H}\,\mathbf{P}(\mathsf{K})$.
(ii) $\mathbf{P}\,\mathbf{S}(\mathsf{K}) \subseteq \mathbf{S}\,\mathbf{P}(\mathsf{K})$.
(iii) $\mathbf{H}\,\mathbf{P}$, $\mathbf{S}\,\mathbf{P}$, *and* $\mathbf{H}\,\mathbf{S}\,\mathbf{P}$ *are closure operators on* $\mathsf{Alg}(\Sigma)$.

Proof. (i) Assume $\boldsymbol{A} \in \mathbf{P}\,\mathbf{H}(\mathsf{K})$. Then there is a $\langle\, \boldsymbol{B}_i : i \in I \,\rangle \in \mathbf{H}(\mathsf{K})^I$ such that $\boldsymbol{A} \cong \prod_{i\in I} \boldsymbol{B}_i$. Let $\langle\, \boldsymbol{C}_i : i \in I \,\rangle \in \mathsf{K}^I$ such that $\boldsymbol{C}_i \succcurlyeq \boldsymbol{B}_i$ for all $i \in I$. For each $i \in I$ let $h_i\colon \boldsymbol{C}_i \twoheadrightarrow \boldsymbol{B}_i$. Then $h_i \circ \pi_i\colon \prod_{i\in I} \boldsymbol{C}_i \twoheadrightarrow \boldsymbol{B}_i$ for each $i \in I$. By the Categorical Product Property there is a unique homomorphism $\bar{h}\colon \prod_{i\in I} \boldsymbol{C}_i \to \prod_{i\in I} \boldsymbol{B}_i$ such that $\pi_i \circ \bar{h} = h_i \circ \pi_i\colon \prod_{i\in I} \boldsymbol{C}_i \to \boldsymbol{B}_i$ for every $i \in I$. We denote $\bar{h}$ by $\prod_{i\in I} h_i$ and refer to it as the *natural map.*

Let $\vec{b} = \langle\, b_i : i \in I \,\rangle \in \prod_{i\in I} \boldsymbol{B}_i$. Choose $\vec{c} = \langle\, c_i : i \in I \,\rangle \in \prod_{i\in I} \boldsymbol{C}_i$ such that $h_i(c_i) = b_i$ for every $i \in I$. Then $\big(\prod_{i\in I} h_i\big)(\vec{c}) = \langle\, h_i(c_i) \colon i \in I \,\rangle = \vec{b}$. Hence $\prod_{i\in I} h_i$ is surjective and thus $\boldsymbol{A} \in \mathbf{I}\,\mathbf{H}\,\mathbf{P}(\mathsf{K}) = \mathbf{H}\,\mathbf{P}(\mathsf{K})$.

(ii) Assume $\boldsymbol{A} \in \mathbf{P}\,\mathbf{S}(\mathsf{K})$. There is a $\langle\, \boldsymbol{B}_i : i \in I \,\rangle \in \mathsf{K}^I$ and a $\langle\, \boldsymbol{C}_i : i \in I \,\rangle \in \mathbf{S}(\mathsf{K})^I$ such that, for every $i \in I$, $\boldsymbol{C}_i \subseteq \boldsymbol{B}_i$ and $\boldsymbol{A} \cong \prod_{i\in I} \boldsymbol{C}_i$. But $\prod_{i\in I} \boldsymbol{C}_i \subseteq \prod_{i\in I} \boldsymbol{B}_i$ (exercise). So $\boldsymbol{A} \in \mathbf{I}\,\mathbf{S}\,\mathbf{P}(\mathsf{K}) = \mathbf{S}\,\mathbf{P}(\mathsf{K})$.

(iii) For every $\mathsf{K} \subseteq \mathsf{Alg}(\Sigma)$, $\mathsf{K} \subseteq \mathbf{P}(\mathsf{K})$, which implies $\mathsf{K} \subseteq \mathbf{S}(\mathsf{K}) \subseteq \mathbf{S}\,\mathbf{P}(\mathsf{K})$, which in turn implies that $\mathsf{K} \subseteq \mathbf{H}(\mathsf{K}) \subseteq \mathbf{H}\,\mathbf{S}(\mathsf{K}) \subseteq \mathbf{H}\,\mathbf{S}\,\mathbf{P}(\mathsf{K})$. Thus

$$\begin{aligned}
\mathbf{H}\,\mathbf{S}\,\underline{\mathbf{P}\,\mathbf{H}}\,\mathbf{S}\,\mathbf{P}(\mathsf{K}) &\subseteq \mathbf{H}\,\mathbf{S}\,\underline{\mathbf{H}\,\mathbf{P}}\,\mathbf{S}\,\mathbf{P}(\mathsf{K})\\
&= \mathbf{H}\,\underline{\mathbf{S}\,\mathbf{H}}\,\mathbf{P}\,\mathbf{S}\,\mathbf{P}(\mathsf{K})\\
&\subseteq \mathbf{H}\,\underline{\mathbf{H}\,\mathbf{S}}\,\mathbf{P}\,\mathbf{S}\,\mathbf{P}(\mathsf{K})\\
&= \mathbf{H}\,\mathbf{H}\,\mathbf{S}\,\underline{\mathbf{P}\,\mathbf{S}}\,\mathbf{P}(\mathsf{K})\\
&= \mathbf{H}\,\mathbf{H}\,\mathbf{S}\,\underline{\mathbf{S}\,\mathbf{P}}\,\mathbf{P}(\mathsf{K})\\
&= \mathbf{H}\,\mathbf{S}\,\mathbf{P}(\mathsf{K}).
\end{aligned}$$

So $\mathbf{HSPHSP}(\mathsf{K}) \subseteq \mathbf{HSP}(\mathsf{K})$; the inclusion in the opposite direction is immediate.

Clearly $\mathsf{K} \subseteq \mathsf{L}$ implies $\mathbf{HSP}(\mathsf{K}) \subseteq \mathbf{HSP}(\mathsf{K})$. □

Corollary 2.60. *Let* K *be any class of* Σ*-algebras. The following conditions are equivalent.*

(i) $\mathbf{H}(\mathsf{K}) \subseteq \mathsf{K}$, $\mathbf{S}(\mathsf{K}) \subseteq \mathsf{K}$, $\mathbf{P}(\mathsf{K}) \subseteq \mathsf{K}$, *i.e.,* K *is closed under the formation of homomorphic images, subalgebras, and isomorphic images of direct products.*

(ii) $\mathbf{HSP}(\mathsf{K}) = \mathsf{K}$.

Proof. (i) $\Longrightarrow$ (ii). $\mathbf{HSP}(\mathsf{K}) \subseteq \mathbf{HS}(\mathsf{K}) \subseteq \mathbf{H}(\mathsf{K}) \subseteq \mathsf{K}$; $\mathsf{K} \subseteq \mathbf{HSP}(\mathsf{K})$ always holds.

(ii) $\Longrightarrow$ (i). $\mathbf{H}(\mathsf{K}) = \mathbf{HHSP}(\mathsf{K}) = \mathbf{HSP}(\mathsf{K}) = \mathsf{K}$. $\mathbf{S}(\mathsf{K}) = \mathbf{SHSP}(\mathsf{K}) \subseteq \mathbf{HSSP}(\mathsf{K}) = \mathbf{HSP}(\mathsf{K}) = \mathsf{K}$. $\mathbf{P}(\mathsf{K}) = \mathbf{PHSP}(\mathsf{K}) \subseteq \mathbf{HPSP}(\mathsf{K}) \subseteq \mathbf{HSPP}(\mathsf{K}) = \mathbf{HSP}(\mathsf{K}) = \mathsf{K}$. □