The Correspondence Theorem, which we next prove, shows that the congruence lattice of every homomorphic image of a Σ-algebra is isomorphically embeddable as a special kind of sublattice of the congruence lattice of the algebra itself. To prepare for the theorem we must describe the special kind of sublattices that are involved.

Let $\boldsymbol{L} = \langle L, \wedge, \vee \rangle$ be a lattice. A subset X of $\boldsymbol{L}$ is a *lower segment* $x \in X$ implies $y \in X$ for every $y \leq x$. An lower segment is an *ideal* of $\boldsymbol{L}$ if it is closed under join, i.e., $x, y \in X$ implies $x \vee y \in X$. An ideal is obviously a subuniverse of $\boldsymbol{L}$, in fact, if $x \in X$ then $x \wedge y \in X$ for every $y \in L$. A *dual ideal* of $\boldsymbol{L}$ is an ideal of the dual lattice, i.e., X is a dual ideal if $x \in X$ implies $y \in X$ for every $y \geq x$ (X is an *upper segment*), and X is closed under meet. A dual ideal of $\boldsymbol{L}$ is also a subuniverse.

For every $a \in L$ we define:

$$L(a] = \{\, x \in L : x \leq a \,\} \quad \text{and} \quad L[a) = \{\, x \in L : a \leq x \,\}.$$

It is easy to see that $L(a]$ is an ideal of $\boldsymbol{L}$ and $L[a)$ is a dual ideal. $L[a)$ and $L(a]$ are subuniverses of $\boldsymbol{L}$ and the corresponding sublattices are denoted by $\boldsymbol{L}[a)$ and $\boldsymbol{L}(a]$, respectively. If $\boldsymbol{L}$ is a complete lattice, then $\boldsymbol{L}[a)$ and $\boldsymbol{L}(a]$ are complete sublattices (exercise). $\boldsymbol{L}(a]$ is called the *principal ideal generated by* a and $\boldsymbol{L}(a]$ is the *principal dual ideal generated by* a. See Figure 9

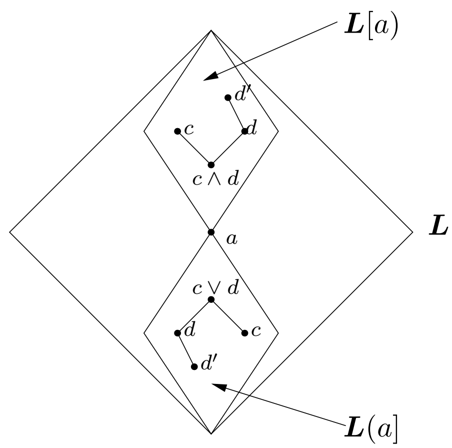

FIGURE 9

For example, in the chain lattice $\langle \mathbb{R}, \vee, \wedge \rangle$ of real numbers under the natural ordering, the principal ideals are the lower infinite closed intervals $(-\infty, a]$ and the principal dual ideals are the upper infinite closed intervals $[a, \infty)$.

Theorem 2.31 (Correspondence Theorem)**.** *Let* $\boldsymbol{A}$ *be a* Σ*-algebra and* γ *a congruence on* $\boldsymbol{A}$*. Then* $\mathbf{Co}(\boldsymbol{A}/\gamma) \cong \mathbf{Co}(\boldsymbol{A})[\gamma)$*. In particular, the mapping* $\alpha \mapsto \alpha/\gamma$ *is an isomorphism from* $\mathbf{Co}(\boldsymbol{A})[\gamma)$ *to* $\mathbf{Co}(\boldsymbol{A}/\gamma)$*, where* $\alpha/\gamma = \{\, \langle a/\gamma, a'/\gamma \rangle : a \,\alpha\, a' \,\}$*.*

Proof. For $\alpha \in \mathrm{Co}[\gamma)$ we have $\alpha/\gamma = \{\, \langle \Delta_\gamma(a), \Delta_\gamma(a) \rangle : a \,\alpha\, a' \,\} = \Delta_\gamma(\alpha)$, where $\Delta_\gamma \colon \boldsymbol{A} \twoheadrightarrow \boldsymbol{A}/\gamma$ is the natural map. So we must show that $\Delta_\gamma \colon \mathbf{Co}(\boldsymbol{A})[\gamma) \to \mathbf{Co}(\boldsymbol{A}/\gamma)$ is a lattice isomorphism. Note that $\mathrm{rker}(\Delta_\gamma) = \gamma$, because $\Delta_\gamma(a) = \Delta_\gamma(a')$ iff $a/\gamma = a'/\gamma$ iff $a \equiv_\gamma a'$.

Let $\beta \in \mathrm{Co}(\boldsymbol{A}/\gamma)$. By Lem. 2.30(i) we have $\gamma \subseteq \Delta_\gamma^{-1}(\beta) \in \mathrm{Co}(\boldsymbol{A})$, i.e., $\Delta_\gamma^{-1}(\beta) \in \mathrm{Co}(\boldsymbol{A})[\gamma)$. $\Delta_\gamma \Delta_\gamma^{-1}(\beta) = \beta$ by set theory since the mapping $\langle a, a' \rangle \mapsto \langle a/\gamma, a'/\gamma \rangle$ from A^2 to $(A/\gamma)^2$ is surjective.

Let $\alpha \in \mathrm{Co}(\boldsymbol{A})[\gamma)$, i.e., $\gamma \subseteq \alpha \in \mathrm{Co}(\boldsymbol{A})$. Then $\Delta_\gamma(\alpha) \in \mathrm{Co}(\boldsymbol{A}/\gamma)$ by Lem. 2.30(ii) because the relation kernel of Δ_γ, namely γ, is included in α. We claim that $\Delta_\gamma^{-1}\Delta_\gamma(\alpha) = \alpha$. By set theory $\alpha \subseteq \Delta_\gamma^{-1}\Delta_\gamma(\alpha)$. For the opposite inclusion, suppose $a \equiv a' \left(\Delta_\gamma^{-1}\Delta_\gamma(\alpha)\right)$. Then $\Delta_\gamma(a) \equiv \Delta_\gamma(a') \left(\Delta_\gamma \Delta_\gamma^{-1}\Delta_\gamma(\alpha)\right)$. But $\Delta_\gamma(a) = a/\gamma$ and $\Delta_\gamma(a') = a/\gamma$, and $\Delta_\gamma \Delta_\gamma^{-1}\Delta_\gamma(\alpha) \subseteq \Delta_\gamma(\alpha)$ by set theory. So $a/\gamma \equiv_{\Delta_\gamma(\alpha)} a'/\gamma$. Thus there exist $a_0, a_0' \in A$ such that $a/\gamma = a_0/\gamma$ and $a'/\gamma = a_0'/\gamma$ (i.e., $a_0 \equiv_\gamma a$ and $a_0' \equiv_\gamma a'$) and $a_0 \equiv_\alpha a_0'$. Since $\gamma \subseteq \alpha$ we have $a \equiv_\alpha a'$. Thus $\Delta_\gamma^{-1}\Delta_\gamma(\alpha) = \alpha$. Hence $\Delta_\gamma \colon \mathrm{Co}(\boldsymbol{A})[\gamma) \twoheadrightarrow \mathrm{Co}(\boldsymbol{A}/\gamma)$ is a bijection with inverse Δ_γ^{-1}. $\alpha \subseteq \alpha'$ implies $\Delta_\gamma(\alpha) \subseteq \Delta_\gamma(\alpha')$ which in turn implies $\alpha = \Delta_\gamma^{-1}\Delta_\gamma(\alpha) \subseteq \Delta_\gamma^{-1}\Delta_\gamma(\alpha') = \alpha'$. So Δ_γ is strictly order-preserving and hence a lattice isomorphism by Thm. 1.8. □

As an example we consider the 3-atom Boolean algebra $\boldsymbol{B}_3$ (or more precisely its lattice reduct). Let h be the endomorphism of $\boldsymbol{B}_3$ indicated by the arrows in Figure 10 and let α be its relation kernel.

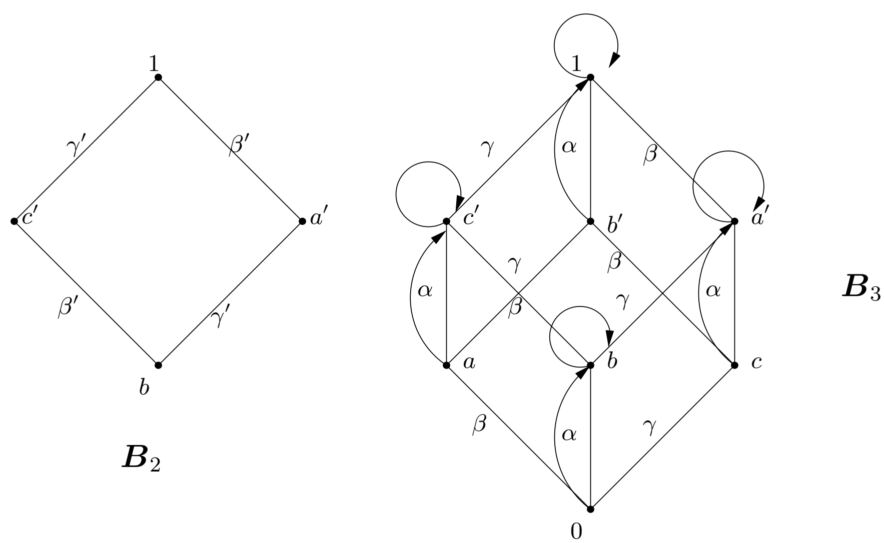

Figure 10

α is indicated on the Hasse diagram for $\boldsymbol{B}_3$ by labeling the edge between every pair of elements congruent mod α with "α". The two proper congruences β and γ that properly include α are also indicated on the diagram. β and γ are respectively he relation kernels of the endomorphisms that push the right front face onto the back left face, and the left front face onto the back right face. The quotient lattice $\boldsymbol{B}_3/\alpha$ is isomorphic to the image of $\boldsymbol{B}_3$ under h which is the 2-atom Boolean algebra $\boldsymbol{B}_2$. As guaranteed by the Correspondence Theorem there are just two proper congruences of $\boldsymbol{B}_2$ corresponding respectively to the β and γ; moreover $\mathbf{Co}(\boldsymbol{B}_2)$ and $\mathbf{Co}(\boldsymbol{B}_3)[\alpha)$ are isomorphic. See Figure 11.

In general, to form the join of a pair of congruences in the lattice of congruences (and also in the lattice of equivalence relations) one has to take arbitrary long alternating relative products of the two congruences (see Theorem 1.17, and the exercise immediately following

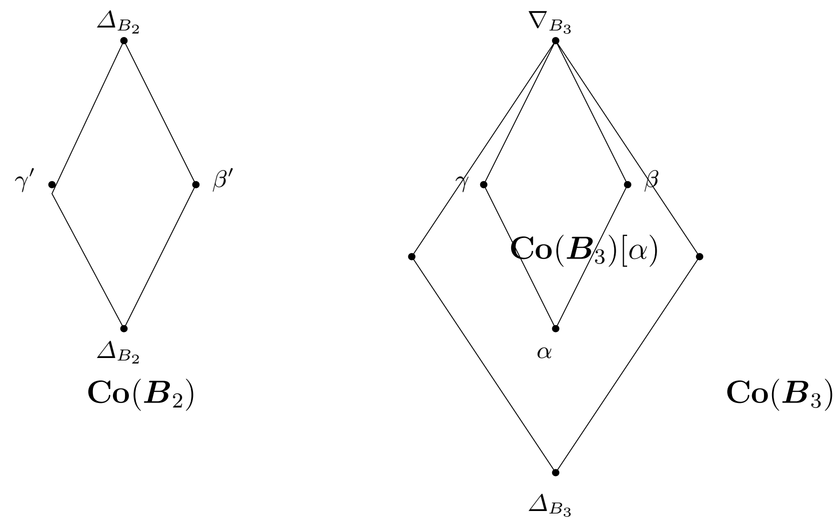

FIGURE 11

it, together with Theorem 2.29). For an important class of algebras only one iteration of the relative product is needed.

Theorem 2.32. *Let $\boldsymbol{A}$ be a Σ-algebra and let $\alpha, \beta \in \mathrm{Co}(\boldsymbol{A})$. The following are equivalent.*

(i) $\alpha \mathbin{;} \beta \subseteq \beta \mathbin{;} \alpha$.
(ii) $\alpha \mathbin{;} \beta = \beta \mathbin{;} \alpha$.
(iii) $\alpha \mathbin{;} \beta \in \mathrm{Co}(\boldsymbol{A})$.
(iv) $\alpha \mathbin{;} \beta = \alpha \vee^{\mathbf{Co}(\boldsymbol{A})} \beta$.

Proof. Here is the chain of implications we will prove: (i) $\Longleftrightarrow$ (ii) $\Longrightarrow$ (iv) $\Longleftrightarrow$ (iii) $\Longrightarrow$ (ii).

(i) $\Longleftrightarrow$ (ii). The implication from right to left is trivial. Assume $\alpha;\beta \subseteq \beta;\alpha$. Since α and β are symmetric, $\breve{\alpha} = \alpha$ and $\breve{\beta} = \beta$. Thus $\beta \mathbin{;} \alpha = \breve{\beta} \mathbin{;} \breve{\alpha} = (\alpha \mathbin{;} \beta)^{\smile} \subseteq (\beta \mathbin{;} \alpha)^{\smile} = \breve{\alpha} \mathbin{;} \breve{\beta} = \alpha \mathbin{;} \beta$.

(i) $\Longrightarrow$ (iv). Assume $\alpha \mathbin{;} \beta = \beta \mathbin{;} \alpha$. By Theorems 1.17 and 2.29 we have

$$\alpha \vee^{\mathbf{Co}(\boldsymbol{A})} \beta = \alpha \vee^{\mathbf{Eq}(\boldsymbol{A})} \beta = \alpha \mathbin{;} \beta \cup \alpha \mathbin{;} \beta \mathbin{;} \alpha \mathbin{;} \beta \cup \cdots \cup (\alpha \mathbin{;} \beta)^n \cup \cdots .$$

The claim is that $(\alpha \mathbin{;} \beta)^n = \alpha \mathbin{;} \beta$ for all $n \in \omega$. This is trivial if $n = 1$. $(\alpha \mathbin{;} \beta)^{n+1} = (\alpha \mathbin{;} \beta)^n \alpha \mathbin{;} \beta \underset{\text{ind. hyp.}}{=} \alpha \mathbin{;} \beta \mathbin{;} \alpha \mathbin{;} \beta \underset{(ii)}{=} \alpha \mathbin{;} \alpha \mathbin{;} \beta \mathbin{;} \beta = \alpha \mathbin{;} \beta$. So $\alpha \vee^{\mathbf{Co}(\boldsymbol{A})} \beta = \alpha \mathbin{;} \beta$.

(iii) $\Longleftrightarrow$ (iv). Obvious.

(iii) $\Longrightarrow$ (ii). Assume $\alpha \mathbin{;} \beta \in \mathrm{Co}(\boldsymbol{A})$. Then $\alpha \mathbin{;} \beta = (\alpha \mathbin{;} \beta)^{\smile} = \breve{\beta} \mathbin{;} \breve{\alpha} = \beta \mathbin{;} \alpha$. □

Definition 2.33. A Σ-algebra $\boldsymbol{A}$ has *permutable congruence relations* if, for all $\alpha, \beta \in \mathrm{Co}(\boldsymbol{A})$, $\alpha \mathbin{;} \beta = \beta \mathbin{;} \alpha$.

Theorem 2.34. *Every group has permutable congruence relations.*

Proof. Let $\boldsymbol{G} = \langle G, \cdot, -1, e \rangle$ be a group and $\alpha, \beta \in \mathrm{Co}(\boldsymbol{G})$. Suppose $\langle a, b \rangle \in \alpha \mathbin{;} \beta$. Then there is a $c \in G$ such that $a \,\alpha\, c \,\beta\, b$. The claim is that

$$a = ac^{-1}c \equiv_\beta ac^{-1}b \equiv_\alpha cc^{-1}b = b.$$

$ac^{-1} \equiv_\beta ac^{-1}$ and $c \equiv_\beta b$ imply $(ac^{-1})c \equiv_\beta (ac^{-1})b$, and $a \equiv_\alpha c$ and $c^{-1}b \equiv_\alpha c^{-1}b$ imply $a(c^{-1}b) \equiv_\alpha c(c^{-1}b)$. So $\langle a, b\rangle \in \beta \mathbin{;} \alpha$. Hence $\alpha \mathbin{;} \beta \subseteq \beta \mathbin{;} \alpha$. □

The permutability of congruence relations on groups is a reflection of the fact that normal subuniverses permute under complex product, and hence that the relative product of two normal subuniverses is a normal subuniverse (recall the correspondence between congruences and normal subuniverses). This property of normal subgroups implies the following important property of groups. For all normal subuniverses N, M, and Q of a group $\boldsymbol{G}$,

$$N \subseteq M \text{ implies } N(M \cap Q) = M \cap (NQ).$$

This is the modular law for the lattice of normal subuniverses.

Definition 2.35. A lattice is *modular* if it satisfies the quasi-identity

$$\forall x, y\,\big(x \le y \to x \vee (y \wedge z) \approx y \wedge (x \vee z)\big). \tag{23}$$

Remarks:

(1) The inclusion $x \vee (y \wedge z) \le y \wedge (x \vee z)$ always holds if $x \le y$ because $x \le y$ and $x \le x \vee z$ imply $x \le y \wedge (x \vee z)$, and $y \wedge z \le y$ and $y \wedge z \le z \le x \vee z$ imply $y \wedge z \le y \wedge (x \vee z)$. And $x \le y \wedge (x \vee z)$ and $y \wedge z \le y \wedge (x \vee z)$ together imply $x \vee (y \wedge z) \le y \wedge (x \vee z)$.

(2) The quasi-identity (23) is equivalent to the quasi-identity (by custom we omit the explicit quantifier)

$$(x \wedge y) \approx x \to x \vee (y \wedge z) \approx y \wedge (x \vee z)\big),$$

and hence to the identity obtained by substituting $x \wedge y$ for x in the right-hand side of this quasi-identity:

$$(x \wedge y) \vee (y \wedge z) \approx \big(\underbrace{(x \wedge y) \vee y}_{y}\big) \wedge \big((x \wedge y) \vee z\big).$$

So every distributive lattice is modular, but not conversely. The 3-atom distributive lattice $\boldsymbol{M}_3$ (see Figure 12) is modular but not conversely.

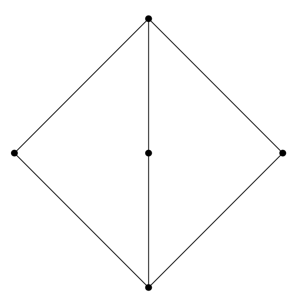

Figure 12

There is easy way to see that a lattice is modular from its Hasse diagram. Let $\boldsymbol{L}$ be an arbitrary lattice and let x, y, z be arbitrary elements such that $x \le y$. If the equation on the right side of (23) fails to hold, it is a easy to see that sub-poset of $\boldsymbol{L}$ with elements $x, y, z, x \vee z, y \wedge z, y \wedge (x \vee z)$, and $x \vee (y \wedge z)$ has the Hasse diagram given in Figure 13. Hence if a lattice fails to satisfy the modular law, then it must contain the lattice in Figure 14 as a sublattice.

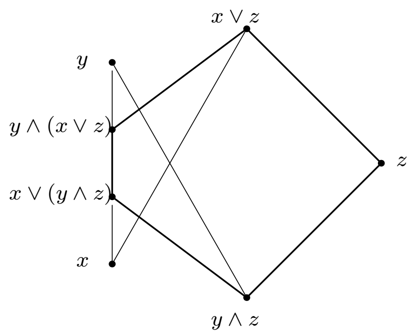

FIGURE 13

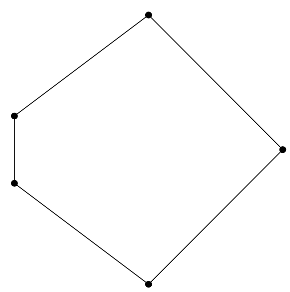

FIGURE 14

This is the lattice $\boldsymbol{N}_5$. It is clearly nonmodular and from the above analysis we see that an arbitrary lattice is nonmodular iff it includes $\boldsymbol{N}_5$ as a sublattice. This simple characterization of nonmodular lattices is due to Dedekind. It follows immediately from this that $\boldsymbol{D}_3$ is modular.

Dedekind was also the one to show that the lattice of normal subgroups of a a lattice is modular. We will generalize this result to show that any congruence-permutable Σ-algebra has a modular congruence lattice. It turns our that there is more general form of the modular law that holds for the binary relations on any set.

Lemma 2.36 (Generalized Modular Law for Relations)**.** *Let A be any nonempty set and let $\alpha \in \mathrm{Eq}(A)$ and $\beta, \gamma \subseteq A \times A$.*

$$\beta \subseteq \alpha \textit{ implies } \beta\,;(\alpha \cap \beta)\,;\beta = \alpha \cap (\beta\,;\gamma\,;\beta). \tag{24}$$

Proof. Assume $\beta \subseteq \alpha$. Then $\beta;(\alpha\cap\gamma);\beta \subseteq \alpha;\alpha;\alpha = \alpha$. Since the inclusion $\beta;(\alpha\cap\gamma);\beta \subseteq \beta\,;\gamma\,;\beta$ is obvious, we have the inclusion from left to right on the right-hand side of (24). So it suffices to prove

$$\alpha \cap (\beta\,;\gamma\,;\beta) \subseteq \beta\,;(\alpha \cap \beta)\,;\beta.$$

Let $\langle x, y\rangle \in \alpha \cap (\beta\,;\gamma\,;\beta)$, and let $a, b \in A$ such that $x\,\beta\,a\,\gamma\,b\,\beta\,y$. Consider the diagram in Figure 15.

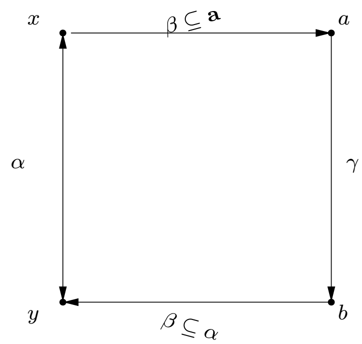

FIGURE 15

There are two ways to get from a to b. One is directly along the γ-arrow. The other is back along the reverse of the β-arrow to x and and then along the α-arrow to y and then along the reverse of the β-arrow to b. This gives $a\,(\breve{\beta}\,;\alpha\,;\breve{\beta})\,b$. But $\breve{\beta}\,;\alpha\,;\breve{\beta} \underset{\beta\subseteq\alpha}{\subseteq} \breve{\alpha}\,;\alpha\,;\breve{\alpha} = \alpha$, since α is assumed to be an equivalence relation. So $a\,(\alpha\cap\gamma)\,b$, and hence $x\,\beta\,a\,(\alpha\cap\gamma)\,b\,\beta\,y$. So $\langle x, y\rangle \in \beta\,;(\alpha\cap\gamma)\,\beta$. □

Theorem 2.37. *Let* $\boldsymbol{A}$ *be a* Σ*-algebra. If* $\boldsymbol{A}$ *has permutable congruence relations, then* $\mathbf{Co}(\boldsymbol{A})$ *is modular.*

Proof. Let $\alpha, \beta, \gamma \in \mathrm{Co}(\boldsymbol{A})$. $\beta\,;\gamma\,;\beta = \beta\,;\beta\,;\gamma = \beta\,;\gamma = \beta \vee \alpha$. And $\beta\,;(\alpha\cap\gamma)\,;\beta = \beta\,;\beta\,;(\alpha\cap\gamma) = \beta\,;(\alpha\cap\gamma) = \beta\vee(\alpha\cap\gamma)$. Thus, by Lem. 2.36,

$$\beta \subseteq \alpha \text{ implies } \beta\vee(\alpha\cap\gamma) = \alpha\cap(\beta\vee)\gamma).$$

□

2.7. **Simple algebras.** A common theme in algebra is to analyze the structure of a complex algebra by attempting to decompose it in some regular way into simpler algebras. The simplest groups from this point of groups are the groups that have no nonisomorphic nontrivial homomorphic images. These are the so-called simple groups and they have natural generalization to arbitrary Σ-algebras.

Definition 2.38. A Σ-algebra $\boldsymbol{A}$ is *simple* if it is nontrivial ($|A| \geq 2$) and $\mathrm{Co}\,\boldsymbol{A} = \{\Delta_A, \nabla_A\}$. Equivalently, $\boldsymbol{A}$ is *simple* if it is nontrivial and, for every Σ-algebra $\boldsymbol{B}$ and every epimorphism from $\boldsymbol{A}$ to $\boldsymbol{B}$, either $|B| = 1$ or $h\colon \boldsymbol{A} \cong \boldsymbol{B}$.

Remarks:

(1) Let $\alpha \in \mathrm{Co}(\boldsymbol{A})$. By the Correspondence Theorem $\boldsymbol{A}/\alpha$ is simple iff α is a coatom of the congruence lattice $\mathbf{Co}(\boldsymbol{A})$, i.e., α is a maximal proper congruence.

(2) $\boldsymbol{A}$ is simple iff $|A| \geq 2$ and $\Theta_{\boldsymbol{A}}(a, b) = \nabla_A$ every pair a, b of distinct elements of $\boldsymbol{A}$.

Examples:

(1) A group is simple iff it has no normal subuniverses. The alternating group A_n is simple for every $n \geq 5$.

(2) The only simple Abelian groups are $\mathbb{Z}_p$ for each prime p.

(3) Every field $\langle R, +, \cdot, -, 0, 1\rangle$ is simple.

(4) The n-atom modular lattice $\boldsymbol{M}_n$ is simple for every $n \geq 3$. See Figure 16.

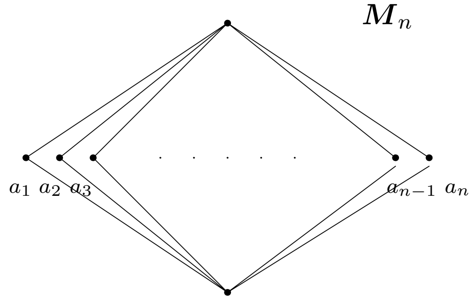

FIGURE 16

To see this first let $\alpha = \Theta_{\boldsymbol{M}_n}(a_i, a_j)$ with $i \neq j$. $1 = a_i \vee a_j \equiv_\alpha a_i \vee a_i = a_i = a_i \wedge a_i \equiv_\alpha a_i \wedge a_j = 0$. So $\Theta_{\boldsymbol{M}_n}(a_i, a_j) = \nabla_{M_n}$. Now let $\alpha = \Theta_{\boldsymbol{M}_n}(a_i, 0)$ and choose any $j \neq i$. $1 = a_i \vee a_j \equiv_\alpha 0 \vee a_j = a_j$. So $\Theta_{\boldsymbol{M}_n}(a_j, 1) \subseteq \Theta_{\boldsymbol{M}_n}(a_i, 0)$ for all $j \neq i$. If i, j, k are all distinct, then $a_j \equiv_\alpha 1 \equiv_\alpha a_k$, and hence $\nabla_{M_n} = \Theta_{\boldsymbol{M}_n}(a_j, a_k) \subseteq \Theta_{\boldsymbol{M}_n}(a_i, 0)$. Similarly $\Theta_{\boldsymbol{M}_n}(a_i, 1) = \nabla_{M_n}$.

(5) The only simple mono-unary algebras are cycles of prime order (exercise).

The proof of the following theorem is also left as an exercise.

Theorem 2.39. *Let $\boldsymbol{A}$ be a nontrivial Σ-algebra.*

(i) *If Δ_A finitely generated as a congruence of $\boldsymbol{A}$, then there exists a simple Σ-algebra $\boldsymbol{B}$ such that $\boldsymbol{B} \preccurlyeq \boldsymbol{A}$.*

(ii) *If Σ is finite (i.e., it has only a finite number of operation symbols) and A is finitely generated as a subuniverse of itself, then there exists a simple Σ-algebra $\boldsymbol{B}$ such that $\boldsymbol{B} \preccurlyeq \boldsymbol{A}$.*

Under the hypotheses of (ii) it can be shown that Δ_A is finitely generated.

As a corollary of this theorem every finite nontrivial Σ-algebra has a simple homomorphic image.

Lemma 2.40. *Let $\boldsymbol{A}$ be a nontrivial Σ-algebra. If $\boldsymbol{A}$ is nonsimple, then $\boldsymbol{A}$ has a nonsimple subalgebra that is generated by at most four elements.*

Proof. Suppose $\boldsymbol{A}$ is nonsimple. Then there exist $a, b \in A$, $a \neq b$, such that $\Theta_{\boldsymbol{A}}(a, b) \neq \nabla_A$. Let $c, d \in A$ such that $\langle c, d \rangle \notin \Theta_{\boldsymbol{A}}(a, b)$, and let $\boldsymbol{B} = \mathbf{Sg}^{\boldsymbol{A}}\{(a, b, c, d\}$ and $\alpha = \Theta_{\boldsymbol{A}}(a, b) \cap B^2$. $\alpha \in \mathrm{Co}(\boldsymbol{B})$, $\langle a, b \rangle \in \alpha$, and $\langle c, d \rangle \neq \alpha$. So $\alpha \neq \Delta_B, \nabla_B$. Hence $\boldsymbol{B}$ is not simple. □

As an immediate consequence of the lemma we have:

Theorem 2.41. *If every finitely generated subalgebra of a Σ-algebra $\boldsymbol{A}$ is simple, then so is $\boldsymbol{A}$.* □

Let Σ be a multi-sorted signature with sort set S, and let $\boldsymbol{A}$ and $\boldsymbol{B}$ be Σ-algebras. A *homomorphism* h for $\boldsymbol{A}$ to $\boldsymbol{B}$ is a S-sorted set $\langle\, h_s : s \in S\,\rangle$ such that $h_s\colon A_s \to B_s$ for every $s \in S$ and such that, for every $\sigma \in \Sigma$ with type $s_1, \dots, s_n \to s$ and all $\langle a_1, \dots, a_n \rangle \in A_{s_1} \times \cdots \times A_{s_n}$,

$$h_s\big(\sigma^{\boldsymbol{A}}(a_1, \dots, a_n)\big) = \sigma^{\boldsymbol{B}}\big(h_{s_1}(a_1), \dots, h_{s_n}(a_n)\big).$$

A *congruence* α on $\boldsymbol{A}$ is an S-sorted set $\langle\, \alpha_s : s \in S \,\rangle$ such that $\alpha_s \in \mathrm{Eq}(A_s)$ for each $s \in S$, and for every $\sigma \in \Sigma$ with type $s_1, \ldots, s_n \to s$ and all $\langle a_1, \ldots, a_n \rangle, \langle b_1, \ldots, b_n \rangle \in A_{s_1} \times \cdots \times A_{s_n}$,

$$\forall\, i \leq n\, (a_i \equiv_{\alpha_{s_i}} b_i) \Longrightarrow \sigma^{\boldsymbol{A}}(a_1, \ldots, a_n) \equiv_{\alpha_s} \sigma^{\boldsymbol{A}}(b_1, \ldots, b_n).$$