(4) Let $\boldsymbol{L} = \langle L, \vee, \wedge, 0, 1\rangle$ be a bounded chain. Let $\alpha \in \mathrm{Co}(\boldsymbol{L})$. If $a\,\alpha\,b$ then, for every $c \in [a,b]$ $(= \{\, x \in L : a \leq x \leq b \,\})$ we have $c = (a \vee c)\,\alpha\,(b \vee c) = b$. So, if any two elements of $\boldsymbol{L}$ are identified, so are all the elements in the interval between the two. This implies that A/α is a partition of L into intervals, either closed, open, or half-open, and it is easy to check that every such partition is the partition of a congruence. For example $\{[0, 1/2), [1/2, 3/4], (3/4, 4/5), [4/5, 1]\}$ is the partition of a congruence of $\langle [0, 1], \leq\rangle$.

Theorem 2.20. *Let $\boldsymbol{A}$ be a Σ-algebra and let $\alpha \in \mathrm{Co}(\boldsymbol{A})$. The mapping $\Delta_\alpha : A \to A/\alpha$ such that, for all $a \in A$, $\Delta_\alpha(a) = a/\alpha$ is an epimorphism from $\boldsymbol{A}$ onto $\boldsymbol{A}/\alpha$ called the* ***natural map****.*

Proof.

$$\begin{aligned}\Delta_\alpha\big(\sigma^{\boldsymbol{A}}(a_1, \dots, a_n)\big) &= \sigma^{\boldsymbol{A}}(a_1, \dots, a_n)/\alpha \\ &= \sigma^{\boldsymbol{A}/\alpha}(a_1/\alpha, \dots, a_n/\alpha) \\ &= \sigma^{\boldsymbol{A}/\alpha}\big(\Delta_\alpha(a_1), \dots, \Delta_\alpha(a_n)\big).\end{aligned}$$

Δ_α is obviously surjective. □

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be Σ-algebras and $h \in \mathrm{Hom}(\boldsymbol{A}, \boldsymbol{B})$. Thinking of h as is graph, a binary relation between A and B, we can form its converse $\breve{h} = \{\, \langle b, a\rangle \in B \times A : h(a) = b \,\}$. Note that $h\,;\breve{h} \subseteq A^2$ and that $a\,(h\,;\breve{h})\,a'$ iff there is a $b \in B$ such that $a\,h\,b\,\breve{h}\,a'$ iff there is a $b \in B$ such that $h(a) = b = h(a')$. Thus

$$h\,;\breve{h} = \{\, \langle a, a'\rangle \in A^2 : h(a) = h(a') \,\}.$$

We call this the *relation kernel of* h, in symbols $\mathrm{rker}(h)$. It is easy to check that $\mathrm{rker}(h)$ is an equivalence relation on A; its associated partition is $\{\, h^{-1}(b) : b \in h(A) \,\}$, where $h^{-1}(b) = \{\, a \in A : h(a) = b \,\}$. The substitution property also holds. In fact, if $h(a_i) = h(a'_i)$, $i \leq n$, then $h\big(\sigma^{\boldsymbol{A}}(a_1, \dots, a_n)\big) = \sigma^{\boldsymbol{B}}\big(h(a_1), \dots, h(a_n)\big) = \sigma^{\boldsymbol{B}}\big(h(a'_1), \dots, h(a'_n)\big) = h\big(\sigma^{\boldsymbol{A}}(a'_1, \dots, a'_n)\big)$. So $\mathrm{rker}(h) \in \mathrm{Co}(\boldsymbol{A})$ for every Σ-algebra $\boldsymbol{B}$ and every $h \in \mathrm{Hom}(\boldsymbol{A}, \boldsymbol{B})$.

The following observation is helpful in understanding the following Homomorphism Theorem. If $\alpha, \beta \in \mathrm{Co}(\boldsymbol{A})$, then $\alpha \subseteq \beta$ iff, for every $a \in A$, $a/\alpha \subseteq a/\beta$; see Figure 3.

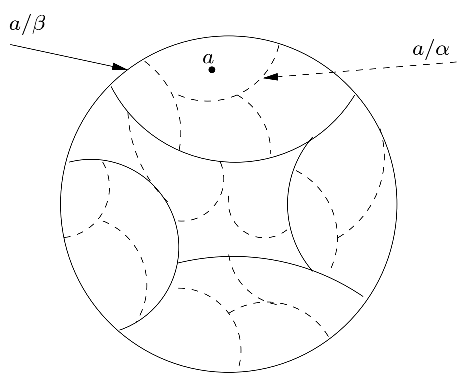

FIGURE 3

Theorem 2.21 (Homomorphism Theorem)**.** *Assume* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *are* Σ*-algebras. Let* $h \in \operatorname{Hom}(\boldsymbol{A},\boldsymbol{B})$*. If* $\alpha \in \operatorname{Co}(\boldsymbol{A})$ *and* $\alpha \subseteq \operatorname{rker}(h)$*, then the map* $h_\alpha \colon A/\alpha \to B$ *defined by* $h_\alpha(a/\alpha) = h(a)$ *for all* $a \in A$ *is well-defined and* $h_\alpha \in \operatorname{Hom}(\boldsymbol{A}/\alpha,\boldsymbol{B})$*. Furthermore,* $h_\alpha \circ \Delta_\alpha = h$*, i.e., the diagram in Figure 4 "commutes" in the sense that either of the two possible paths from* $\boldsymbol{A}$ *to* $\boldsymbol{B}$ *give the same result.*

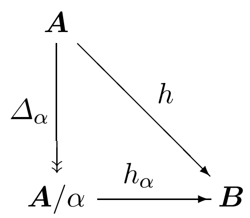

FIGURE 4

Proof. Let $a, a' \in A$ such that $a/\alpha = a'/\alpha$. Then $a/\operatorname{rker}(h) = a'/rker(h)$ since $\alpha \subseteq \operatorname{rker}(h)$. Thus $h(a) = h(a')$. So h_α is well-defined.

$$\begin{aligned} h_\alpha\big(\sigma^{\boldsymbol{A}/\alpha}(a_1/\alpha,\dots,a_n/\alpha)\big) &= h_\alpha(\sigma^{\boldsymbol{A}}(a_1,\dots,a_n)/\alpha\big), && \text{by defn. of } \boldsymbol{A}/\alpha \\ &= h\big(\sigma^{\boldsymbol{A}}(a_1,\dots,a_n)\big), && \text{by defn. of } h \\ &= \sigma^{\boldsymbol{B}}\big(h(a_1),\dots,h(a_n)\big), && \text{since } h \text{ is a homomorphism} \\ &= \sigma^{\boldsymbol{B}}\big(h_\alpha(a_1/\alpha),\dots,h_\alpha(a_n/\alpha)\big), && \text{by defn. of } h. \end{aligned}$$

Thus $h_\alpha \in \operatorname{Hom}(\boldsymbol{A}/\alpha,\boldsymbol{B})$.

Finally we have $(h_\alpha \circ \Delta_\alpha)(a) = h_\alpha\big(\Delta_\alpha(a)\big) = h_\alpha(a/\alpha) = h(a)$. See Figure 5.

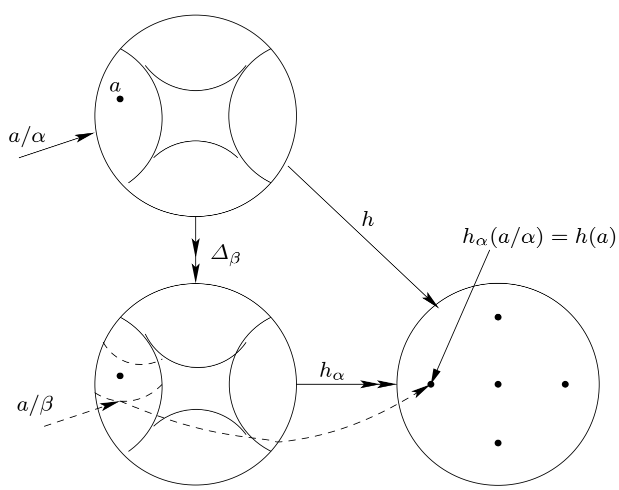

FIGURE 5

Thus Figure 4 commutes. □

On paraphases the conclusion of the theorem by saying that h "factors through" the natural map Δ_α.

Corollary 2.22 (First Isomorphism Theorem)**.** *$\boldsymbol{A} \succcurlyeq B$ iff there exists an $\alpha \in \mathrm{Co}(\boldsymbol{A})$ such that $\boldsymbol{A}/\alpha \cong \boldsymbol{B}$. In particular, if $h\colon \boldsymbol{A} \twoheadrightarrow \boldsymbol{B}$, then $\boldsymbol{A}/\operatorname{rker}(h) \cong \boldsymbol{B}$.*

Proof. Let $h\colon \boldsymbol{A} \twoheadrightarrow \boldsymbol{B}$, and let $\alpha = \operatorname{rker}(h)$. See Figure 6. $h_\alpha(a/\alpha) = h_\alpha(a'/\alpha)$ iff $h(a) =$

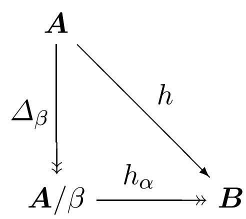

FIGURE 6

$h(a')$ iff $a/\operatorname{rker}(h) = a'/\operatorname{rker}(h)$ iff $a/\alpha = a'/\alpha$. So h_α is injective. Since it is clearly surjective we have $h_\alpha\colon \boldsymbol{A}/\alpha \cong \boldsymbol{B}$. □

Corollary 2.23. *Let $\alpha, \beta \in \mathrm{Co}(\boldsymbol{A})$. $\alpha \subseteq \beta$ implies $\boldsymbol{A}/\alpha/ \succcurlyeq \boldsymbol{A}/gb$.*

Proof. Exercise. □

Theorem 2.24 (Second Isomorphism Theorem)**.** *Let $\boldsymbol{A}$ be a Σ-algebra, and let $\alpha, \beta \in \mathrm{Co}(\boldsymbol{A})$ with $\alpha \subseteq \beta$. Let $\beta/\alpha = \{\, \langle a/\alpha, a'/\alpha \rangle : a \mathrel{\alpha} a' \,\}$. Then $\beta/\alpha \in \mathrm{Co}(\boldsymbol{A}/\alpha)$ and $(\boldsymbol{A}/\alpha)\big/(\beta/\alpha) \cong \boldsymbol{A}/\beta$.*

Proof. By the Homomorphism Theorem $(\Delta_\beta)_\alpha\colon \boldsymbol{A}/\alpha \twoheadrightarrow \boldsymbol{A}/\beta$ where $(\Delta_\beta)_\alpha(a/\alpha) = \Delta_\beta(a) = a/\beta$. See Figure 7.

$$\begin{aligned}
\langle a/\alpha,\, a'/\alpha \rangle \in \operatorname{rker}\big((\Delta_\beta)_\alpha\big) \quad &\text{iff} \quad (\Delta_\beta)_\alpha(a/\alpha) = (\Delta_\beta)_\alpha(a'/\alpha) \\
&\text{iff} \quad a/\beta = a'/\beta \\
&\text{iff} \quad a \mathrel{\beta} a' \\
&\text{iff} \quad \langle a/\alpha,\, a'/\alpha \rangle \in \beta/\alpha.
\end{aligned}$$

So $\operatorname{rker}\big((\Delta_\beta)_\alpha\big) = \beta/\alpha$ and hence $(\boldsymbol{A}/\alpha)\big/(\beta/\alpha) \cong \boldsymbol{A}/\beta$ by the First Isomorphism Theorem. □

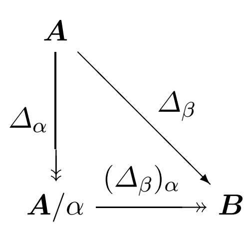

FIGURE 7

Let us recall the Third Isomorphism Theorem for groups. Let H be a subuniverse of a group $\boldsymbol{G}$ and let N be a normal subuniverse of $\boldsymbol{G}$. Let $\boldsymbol{H}\,N$ be the subgroup with universe $H\,N = \{\, h \cdot n : h \in H, n \in N \,\}$. Then $(\boldsymbol{H}\,N)/N \cong \boldsymbol{H}/(H \cap N)$ under the map $h/(H \cap N) \mapsto h/H$.

Let $\boldsymbol{A}$ be a Σ-algebra and let $\boldsymbol{B} \subseteq \boldsymbol{A}$. Let $\alpha \in \mathrm{Co}(\boldsymbol{A})$. Define $B\,\alpha = \bigcup\big(\, b/\alpha : b \in B \,\big\} = \{\, a/\alpha \in A/\alpha : \exists\, b \in B(a\,\alpha\,b\,\}$. Some simple observations:

(1) $\alpha \cap B^2 \in \mathrm{Co}(\boldsymbol{B})$. (exercise)

(2) $B \subseteq B\,\alpha \in \mathrm{Sub}(\boldsymbol{A})$. To see that $B\,\alpha$ is a subuniverse of $\boldsymbol{A}$, we note that $a_1\,\alpha\,b_1, \cdots, a_n\,\alpha\,b_n$ with $b_1, \ldots, b_n \in B$ imply $\sigma^{\boldsymbol{A}}(a_1, \ldots, a_n)\ \alpha\ \sigma^{\boldsymbol{A}}(b_1, \ldots, b_n)$ with $\sigma^{\boldsymbol{A}}(b_1, \ldots, b_n) \in B$.

Let $\boldsymbol{B}\,\alpha$ be the unique subalgebra of $\boldsymbol{A}$ with universe $B\,\alpha$. Then $\boldsymbol{B} \subseteq \boldsymbol{B}\,\alpha \subseteq \boldsymbol{A}$. See Figure 8

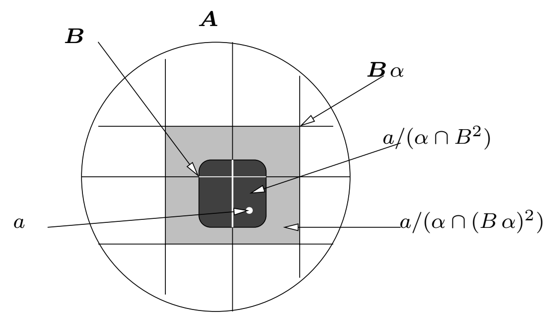

FIGURE 8

Theorem 2.25 (Third Isomorphism Theorem)**.** *Let $\boldsymbol{A}$ be a Σ-algebra, $\boldsymbol{B} \subseteq \boldsymbol{A}$, and $\alpha \in \mathrm{Co}(\boldsymbol{A})$. Then*

$$\boldsymbol{B}\,\alpha\big/(\alpha \cap (B\,\alpha)^2 \cong \boldsymbol{B}\big/(\alpha \cap B^2).$$

Proof. Define $h\colon B \to B\,\alpha\big/\big(\alpha \cap (B\,\alpha)^2\big)$ by $h(b) = b\big/\big(\alpha \cap (B\,\alpha)^2\big)$. h is surjective. Let $\beta = \alpha \cap (B\,\alpha)^2$.

$$\begin{aligned} h\big(\sigma^{\boldsymbol{B}}(b_1, \ldots, b_n)\big) &= \sigma^{\boldsymbol{B}}(b_1, \ldots, b_n)/\beta \\ &= \sigma^{\boldsymbol{B}\,\alpha}(b_1, \ldots, b_n)/\beta \\ &= \sigma^{\boldsymbol{B}\,\alpha/\beta}(b_1/\beta, \ldots, b_n/\beta) \\ &= \sigma^{\boldsymbol{B}\,\alpha/\beta}\big(h(b_1), \ldots, h(b_n)\big). \end{aligned}$$

So $h\colon \boldsymbol{B} \twoheadrightarrow \boldsymbol{B}\,\alpha/\beta$. For all $b, b' \in B$, $h(b) = h(b')$ iff $b/\beta = b'/\beta$ iff $b\,\beta\,b'$ iff $b\ \big(\alpha \cap (B\,\alpha)^2\big)\ b'$ iff $b\ (\alpha \cap B^2)\,b'$ since $b, b' \in B$. So $\mathrm{rker}(h) = \alpha \cap B^2$. Now apply the First Isomorphism Theorem. □

2.6. **The lattice of congruences.** *Notation:* Let $\alpha \in \mathrm{Co}(A)$, more generally, let $\alpha \in \mathrm{Eq}(A)$, i.e., an equivalence relation on A. Then for all $a, b \in A$ the following all mean the same thing. $\langle a, b \rangle \in \alpha$, $\alpha\,\alpha\,\beta$, $a \equiv b \pmod{\alpha}$, $a \equiv b\,(\alpha)$, $a \equiv_\alpha b$.

Theorem 2.26. $\langle A \times A, \mathrm{Co}(\boldsymbol{A}) \rangle$ *is an algebraic closed-set system for every Σ-algebra $\boldsymbol{A}$.*

Proof. Let $\mathcal{K} \subseteq \mathrm{Co}(\boldsymbol{A})$. $\mathcal{K} \subseteq \mathrm{Eq}(A)$, so $\bigcap \mathcal{K} \in \mathrm{Eq}(A)$. We must verify the substitution property. Let $\beta = \bigcap \mathcal{K}$ and assume that $a_i \equiv_\beta b_i$ for all $i \leq n$. Then $a_i \equiv_\alpha b_i$ for all $i \leq n$ and all $\alpha \in \mathcal{K}$. Thus $\sigma^{\boldsymbol{A}}(a_1, \dots, a_n) \equiv_\alpha \sigma^{\boldsymbol{A}}(b_1, \dots, b_n)$ for all $\alpha \in \mathcal{K}$, and hence $\sigma^{\boldsymbol{A}}(a_1, \dots, a_n) \equiv_\beta \sigma^{\boldsymbol{A}}(b_1, \dots, b_n)$. So $\beta \in \mathrm{Co}(\boldsymbol{A})$.

Assume now that $\mathcal{K}$ is directed, and let $\beta = \bigcup \mathcal{K}$. Then $\beta \in \mathrm{Eq}(A)$. Assume $a_i \equiv_\beta b_i$ for all $i \leq n$. Then for each $i \leq n$ there exists a $\alpha_i \in \mathcal{K}$ such that $a_i \equiv_{\alpha_i} b_i$. Take α to be an upper bound in $\mathcal{K}$ for all the α_i, $i \leq n$. α exists because $\mathcal{K}$ is directed. Then $a_i \equiv_\alpha b_i$ for all $i \leq n$. Thus $\sigma^{\boldsymbol{A}}(a_1, \dots, a_n) \equiv_\alpha \sigma^{\boldsymbol{A}}(b_1, \dots, b_n)$, and hence $\sigma^{\boldsymbol{A}}(a_1, \dots, a_n) \equiv_\beta \sigma^{\boldsymbol{A}}(b_1, \dots, b_n)$. So $\bigcup \mathcal{K} \in \mathrm{Co}(\boldsymbol{A})$. □

So $\mathbf{Co}(\boldsymbol{A}) = \langle \mathrm{Co}(\boldsymbol{A}), \vee^{\mathrm{Co}(\boldsymbol{A})}, \cap \rangle$ is a complete lattice where

$$\bigvee\nolimits^{\mathbf{Co}(\boldsymbol{A})} \mathcal{K} = \bigcap \{\, \beta \in \mathrm{Co}(\boldsymbol{A}) : \forall\, \alpha \in \mathcal{K}\, (\alpha \subseteq \beta) \,\}.$$

The associated closure operator $\mathrm{Cl}_{\mathrm{Co}(\boldsymbol{A})}$ gives congruence generation. Thus, for every $X \subseteq A^2$,

$$\mathrm{Cl}_{\mathrm{Co}(\boldsymbol{A})}(X) = \bigcap \{ \alpha \in \mathrm{Co}(\boldsymbol{A}) : X \subseteq \alpha \,\}$$

is the *congruence generated by* X. The traditional notation for this is $\Theta_{\boldsymbol{A}}(X)$, or just $\Theta(X)$ if $\boldsymbol{A}$ is clear from context. If X consists of a single ordered pair, say $X = \{\langle a, b\rangle\}$, then we write $\Theta_{\boldsymbol{A}}(a, b)$ for $\Theta_{\boldsymbol{A}}(X)$. Such a congruence, i.e., one generated by a single ordered pair, is called a *principal congruence*. Principal congruences are the congruence analogs of cyclic subuniverses.

Congruences are special kinds of equivalence relations and both form complete lattices. It is natural to ask about the relationship between the two lattices, in particular if the congruences form a sublattice of the equivalence relations. In fact they form a complete sublattice. In order to prove this the following lemmas about binary relations in general prove useful. We only considered the substitution property for equivalence relations, but the property makes sense for any binary relation on the universe of a Σ-algebra. Let $\boldsymbol{A}$ be a Σ-algebra and let $R \subseteq A^2$. R has the *substitution property* if $\langle a_i, b_i\rangle \in R$ for all $i \leq n$ implies $\langle \sigma^{\boldsymbol{A}}(a_1, \dots, a_n), \sigma^{\boldsymbol{A}}(b_1, \dots, b_n)\rangle \in R$.

Lemma 2.27. *Let* $\boldsymbol{A}$ *be a* Σ*-algebra. If* $R, S \subseteq A^2$ *both have the substitution property, then so does their relative product* $R \,;\, S$*.*

Proof. Suppose $a_i \;(R \,;\, S)\, b_i$ for all $i \leq n$. Then for each $i \leq n$ there is a $c_i \in A$ such that $a_i \,R\, c_i \,S\, b_i$. Thus $\sigma^{\boldsymbol{A}}(a_1, \dots, a_n) \,R\, \sigma^{\boldsymbol{A}}(c_1, \dots, c_n) \,S\, \sigma^{\boldsymbol{A}}(b_1, \dots, b_n)$. Hence $\sigma^{\boldsymbol{A}}(a_1, \dots, a_n)\, R;$ $S\; \sigma^{\boldsymbol{A}}(b_1, \dots, b_n)$. □

Lemma 2.28. *Let* $\boldsymbol{A}$ *be a* Σ*-algebra and let* $\mathcal{R}$ *be a directed set of binary relations on* A*. If each* $R \in \mathcal{R}$ *has the substitution property, then so does* $\bigcup \mathcal{R}$*.*

The proof is straightforward and is left as an exercise.

Theorem 2.29. *Let* $\boldsymbol{A}$ *be a* Σ*-algebra.* $\mathbf{Co}(\boldsymbol{A})$ *is a complete sublattice of* $\mathbf{Eq}(A)$*, i.e., for every* $\mathcal{K} \subseteq \mathrm{Co}(\boldsymbol{A})$*,*

$$\bigvee\nolimits^{\mathbf{Co}(\boldsymbol{A})} \mathcal{K} = \bigvee\nolimits^{\mathbf{Eq}(A)} \mathcal{K}.$$

Proof. The inclusion from right to left holds because $\bigvee^{\mathbf{Co}(\boldsymbol{A})}\mathcal{K}$ is an equivalence relation that includes each congruence in $\mathcal{K}$. For the inclusion in the opposite direction it suffices to show that $\bigvee^{\mathbf{Eq}(A)}\mathcal{K}$ has the substitution property. Let

$$\mathcal{R} = \{\, \alpha_1 \mathbin{;} \alpha_2 \mathbin{;} \cdots \mathbin{;} \alpha_n : n \in \omega, \alpha_1, \dots, \alpha_n \in \mathcal{K} \,\}.$$

$\bigvee^{\mathbf{Eq}(A)}\mathcal{K} = \bigcup \mathcal{R}$ by Thm 1.17 and the exercise following it. Each relation in $\mathcal{R}$ has the substitution property by Lem. 2.27, and hence $\bigvee^{\mathbf{Eq}(A)}\mathcal{K}$ has the substitution property by Lem. 2.28. □

We next prove the analog for congruences of Theorem 2.14 that describes the behavior of subuniverses under homomorphisms and inverse homomorphisms. The situation is more complicated in the case of congruences however. For one thing nice results are obtained only for surjective homomorphisms.

Theorem 2.30. *Let $h\colon \boldsymbol{A} \twoheadrightarrow \boldsymbol{B}$ be an epimorphism between Σ-algebras.*

(i) *For every $\beta \in \mathrm{Co}(\boldsymbol{B})$, $h^{-1}(\beta) := \{\, \langle a, a' \rangle \in A : h(a) \equiv_\beta h(a') \,\} \in \mathrm{Co}(\boldsymbol{A})$, and $\mathrm{rker}(h) \subseteq h^{-1}(\beta)$.*

(ii) *For every $\alpha \in \mathrm{Co}(\boldsymbol{A})$, if $\mathrm{rker}(h) \subseteq \alpha$, then*

$$h(\alpha) := \{\, \langle h(a), h(b,) \rangle : a, a' \in A, a \equiv_\alpha a' \,\} \in \mathrm{Co}(\boldsymbol{B}).$$

(iii) *For every $X \subseteq A^2$, if $\mathrm{rker}(h) \subseteq \Theta_{\boldsymbol{A}}(X)$, then $h\big(\Theta_{\boldsymbol{A}}(X)\big) = \Theta_{\boldsymbol{B}}\big(h(X)\big)$.*

Proof. (i). We have $\boldsymbol{A} \overset{h}{\twoheadrightarrow} \boldsymbol{B} \overset{\Delta_\beta}{\twoheadrightarrow} \boldsymbol{B}/\beta$. Thus $(\Delta_\beta \circ h)\colon \boldsymbol{A} \twoheadrightarrow \boldsymbol{B}/\beta$. $h(a) \equiv_\beta h(a')$ iff $h(a)/\beta = h(a)/\beta$ iff $(\Delta_\beta \circ h)(a) = (\Delta_\beta) \circ h)(a')$. Thus $h^{-1}(\beta) = \mathrm{rker}(\Delta_\beta \circ h) \in \mathrm{Co}(\boldsymbol{A})$. If $h(a) = h(a')$ then obviously $(\Delta_\beta \circ h)(a) = (\Delta_\beta \circ h)(a')$. So $\mathrm{rker}(h) \subseteq \mathrm{rker}(\Delta_\beta \circ h)$ and hence $\mathrm{rker}(\Delta_\beta) \subseteq h^{-1}(\beta)$.

(ii) Assume $\mathrm{rker}(h) \subseteq \alpha$. $h(\Delta_A) = \big\{\, \langle h(a), h(a) \rangle : a \in A \,\big\} = \{\, \langle b, b \rangle : b \in h(A) \,\} = \Delta_{h(A)}$. So $h(\Delta_A) = \Delta_B$ since h is surjective, and hence $\Delta_A \subseteq \alpha$ implies $\Delta_B \subseteq h(\alpha)$. So $h(\alpha)$ is reflexive.

$b \equiv_{h(\alpha)} b'$ implies the existence of $a, a' \in A$ such that $a \equiv_\alpha a'$ and $h(a) = b, h(a') = b'$. But $a' \equiv_\alpha a$, so $b' \equiv_{h(\alpha)} b$. So $h(\alpha)$ is symmetric.

Transitivity requires the assumption $\mathrm{rker}(h) \subseteq \alpha$. Suppose $b_0 \equiv_{h(\alpha)} b_1 \equiv_{h(\alpha)} b_2$. Then there exist $a_0, a_1 \in A$ such that $a_0 \equiv_\alpha a_1$ and $h(a_0) = b_0$ and $h(a_1) = b_1$. There also exist $a_1', a_2 \in A$ such that $a_1' \equiv_\alpha a_2$ and $h(a_1') = b_1$ and $h(a_2) = b_2$. Thus

$$a_0 \equiv_\alpha a_1 \equiv_{\mathrm{rker}(h)} a_1' \equiv_\alpha a_2.$$

Since $\mathrm{rker}(h) \subseteq \alpha$, $a_0 \equiv_\alpha a_2$, and hence $b_0 = h(a_0) \equiv_{h(\alpha)} h(a_2) = b_2$. So $h(\alpha)$ is transitive and hence an equivalence relation.

The proof of the substitution property follows a familar pattern and is represented diagramatically. Assume that for each $i \leq n$ we have

$$\begin{array}{ccc} a_i & \equiv_\alpha & a_i' \\ {\scriptstyle h}\big\downarrow & & {\scriptstyle h}\big\downarrow \\ b_i & \equiv_{h(\alpha)} & b_i'. \end{array}$$

Then we have

$$\begin{array}{ccc}
\sigma^{\boldsymbol{A}}(a_1,\dots,a_n) & \equiv_\alpha & \sigma^{\boldsymbol{A}}(a'_1,\dots,a'_n) \\
\Big\downarrow h & & \Big\downarrow h \\
h\big(\sigma^{\boldsymbol{A}}(a_1,\dots,a_n)\big) & \equiv_{h(\alpha)} & h\big(\sigma^{\boldsymbol{A}}(a'_1,\dots,a'_n)\big) \\
\| & & \| \\
\sigma^{\boldsymbol{B}}(b_1,\dots,b_n) & \equiv_{h(\alpha)} & \sigma^{\boldsymbol{B}}(b'_1,\dots,b'_n).
\end{array}$$

Thus $h(\alpha) \in \mathrm{Co}(\boldsymbol{B})$.

(iii) $h(X) \subseteq h\big(\Theta_{\boldsymbol{A}}(X)\big) \underset{(ii)}{\in} \mathrm{Co}(\boldsymbol{B})$. So $\Theta_{\boldsymbol{B}}\big(h(X)\big) \subseteq h\big(\Theta_{\boldsymbol{A}}(X)\big)$. $X \subseteq h^{-1}\big(\Theta_{\boldsymbol{B}}(h(X))\big) \underset{(i)}{\in} \mathrm{Co}(\boldsymbol{A})$. So $\Theta_{\boldsymbol{A}}(X) \subseteq h^{-1}\big(\Theta_{\boldsymbol{B}}(h(X))\big)$, and hence

$$h\big(\Theta_{\boldsymbol{A}}(X)\big) \subseteq hh^{-1}\big(\Theta_{\boldsymbol{B}}(h(x))\big) \subseteq \Theta_{\boldsymbol{B}}\big(h(X)\big).$$

□