Theorem 2.13. *Let $\boldsymbol{A}, \boldsymbol{B}$ be Σ-algebras and $h \in \mathrm{Hom}(\boldsymbol{A}, \boldsymbol{B})$.*

(i) *For every $K \in \mathrm{Sub}(\boldsymbol{A})$, $h(K) \in \mathrm{Sub}(\boldsymbol{B})$.*

(ii) *For every $L \in \mathrm{Sub}(\boldsymbol{B})$, $h^{-1}(L) := \{\, a \in A : h(a) \in L \,\} \in \mathrm{Sub}(\boldsymbol{A})$.*

(iii) *For every $X \subseteq A$, $h\big(\mathrm{Sg}^{\boldsymbol{A}}(X)\big) \in \mathrm{Sg}^{\boldsymbol{A}}\big(h(X)\big)$.*

Proof. (i). Let $\sigma \in \Sigma_n$ and $b_1, \dots, b_n \in h(K)$. Choose $a_1, \dots, a_n \in K$ such that $h(a_1) = b_1, \dots, h(a_n) = b_n$. Then $\sigma^{\boldsymbol{B}}(b_1, \dots, b_n) = \sigma^{\boldsymbol{B}}\big(h(a_1), \dots, h(a_n)\big) = h\big(\sigma^{\boldsymbol{A}}(a_1, \dots, a_n)\big) \in h(K)$.

(ii). Let $a_1, \dots, a_n \in h^{-1}(L)$, i.e., $h(a_1), \dots, h(a_n) \in L$. Then $h\big(\sigma^{\boldsymbol{A}}(a_1, \dots, a_n)\big) = \sigma^{\boldsymbol{B}}\big(h(a_1), \dots, h(a_n)\big) \in L$. So $\sigma^{\boldsymbol{A}}(a_1, \dots, a_n) \in h^{-1}(L)$.

(iii). $h(X) \subseteq h\big(\mathrm{Sg}^{\boldsymbol{A}}(X)\big) \in \mathrm{Sub}(\boldsymbol{B})$ by part (i). So $\mathrm{Sg}^{\boldsymbol{B}}\big(h(X)\big) \subseteq h\big(\mathrm{Sg}^{\boldsymbol{A}}(X)\big)$. For the reverse inclusion, $X \subseteq h^{-1}(h(X)) \subseteq h^{-1}\big(\mathrm{Sg}^{\boldsymbol{A}}(h(X)\big) \in \mathrm{Sub}(\boldsymbol{A})$ by part (ii). So $\mathrm{Sg}^{\boldsymbol{A}}(X) \subseteq h^{-1}\big(\mathrm{Sg}^{\boldsymbol{A}}(h(X))\big)$. □

$h(\boldsymbol{A})$ will denote the unique subalgebra of $\boldsymbol{B}$ with universe $h(A)$ and if $\boldsymbol{B}' \subseteq \boldsymbol{B}$, then $h^{-1}(\boldsymbol{B}')$ is the unique subalgebra of $\boldsymbol{A}$ with universe $h^{-1}(B')$.

Theorem 2.14. *Let $\boldsymbol{A} = \langle A, f \rangle$ be a finite, cyclic mono-unary algebra with period p and tail length l (see Figure 1). Let $h: \boldsymbol{A} \twoheadrightarrow \boldsymbol{A}'$ be a epimorphism. Then $\boldsymbol{A}'$ is finite, cyclic mono-unary algebra. Let p' be its period and l' its tail length. Then p' divides p and $l' \leq l$.*

Proof. By Theorem 2.13(iii), $\boldsymbol{A}'$ is a cyclic mono-unary algebra, and it is obviously finite. Let $A = \mathrm{Sg}^{\boldsymbol{A}}\big(\{a\}\big)$. Then $A' = \mathrm{Sg}^{\boldsymbol{A}'}(\{h(a)\})$ by Theorem 2.13(iii). By definition p is the smallest $m \in \omega \setminus \{0\}$ such that there is an $n \in \omega$ such that $f^{n+m}(a) = f^n(a)$, and l is the smallest $n \in \omega$ such that $f^{n+p}(a) = f^n(a)$. p' and l' are defined similarly. For every $n \geq l$ and every $q \in \omega$, we have

(20)

$$f^{n+qp}(a) = f^{n-l}\big(f^{l+qp}(a)\big) = f^{n-l}\big(\underbrace{f^p(f^p(\dots(f^p}_{q}(f^l(a)))\dots))\big) = f^{n-l}\big(f^l(a)\big) = f^n(a).$$

We claim that, for all $n, m \in \omega$ with $m > 0$,

$$\text{if } f^{n+m}(a) = f^n(a) \text{ then } p \text{ divides } m.$$

For every $n' \geq n$, $f^{n'+m}(a) = f^{n'-n}\big(f^{n+m}(a)\big) = f^{n'-n}\big(f^n(a)\big) = f^{n'}(a)$. So without loss of generalization we assme $n \geq l$ By the division algorithm, $m = qp + r$ with $0 \leq r < p$. Then by (20), $f^{n+r}(a) = f^{n+r+qp}(a) = f^{n+m}(a) = f^n(a)$. By the minimality of p, $r = 0$; so $p \mid m$.

$f^{l+p}\big(h(a)\big) = h\big(f^{l+p}(a)\big) = h\big(f^l(a)\big) = f^l\big(h(a)\big)$. So by (20) (with $\boldsymbol{A}'$, $h(a)$, and p' in place of $\boldsymbol{A}$, a, and p, respectively), we get that p' divides p. Furthermore, choose q such that $l + qp \geq l'$. Then $f^{l+p'}\big(h(a)\big) = f^{p'}\big(f^l(h(a))\big) = f^{p'}\big(f^{l+qp}(h(a))\big) = f^{l+qp+p'}\big(h(a)\big) = f^{l+qp}\big(h(a)\big) = f^l\big(h(a)\big)$. So $l' \leq l$ by the minimality of l'. □

Define the binary relation $\preccurlyeq$ on $\mathsf{Alg}(\Sigma)$ by $\boldsymbol{A} \preccurlyeq \boldsymbol{B}$ (equivalently $\boldsymbol{B} \succcurlyeq \boldsymbol{A}$) if $\boldsymbol{A}$ is a *homomorphic image* of $\boldsymbol{B}$, i.e., there is an epimorphism $h: \boldsymbol{B} \twoheadrightarrow \boldsymbol{A}$. $\preccurlyeq$ is clearly reflexive and it is also transitive, for if $h: \boldsymbol{B} \twoheadrightarrow \boldsymbol{A}$ and $g: \boldsymbol{C} \twoheadrightarrow \boldsymbol{B}$, then $h \circ g: \boldsymbol{C} \twoheadrightarrow \boldsymbol{A}$. However, $\preccurlyeq$ fails to be antisymmetric in a strong way: in general,

$$\boldsymbol{A} \leq \boldsymbol{B} \text{ and } \boldsymbol{B} \leq \boldsymbol{C} \text{ does not imply } \boldsymbol{A} \cong \boldsymbol{B}.$$

For example, let $\boldsymbol{A} = \langle [0,3], \leq \rangle$ and $\boldsymbol{B} = \langle [0,1] \cup [2,3], \leq \rangle$. Define

$$h(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1, \\ 2 & \text{if } 1 < x < 2, \\ x & \text{if } 2 \leq x \leq 3 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 3x & \text{if } 0 \leq x \leq 1, \\ 3 & \text{if } 1 < x \leq 3. \end{cases}$$

We leave it as an exercise to prove that h is an epimorphism from the lattice $\boldsymbol{A}$ to $\boldsymbol{B}$ and that g is an epimorphism in the opposite direction. However, $\boldsymbol{A} \not\cong \boldsymbol{B}$. To see this note that an isomorphism preserves compact elements, but $\boldsymbol{A}$ had only one compact element, 0, while $\boldsymbol{B}$ has two, 0 and 2.

If $\boldsymbol{A}$ or $\boldsymbol{B}$ is finite, then $\boldsymbol{A} \preccurlyeq \boldsymbol{B}$ and $\boldsymbol{B} \preccurlyeq \boldsymbol{A}$ implies $\boldsymbol{A} \cong \boldsymbol{B}$. Because, $\boldsymbol{A} \preccurlyeq \boldsymbol{B}$ and $\boldsymbol{B} \preccurlyeq \boldsymbol{A}$ imply $|A| \leq |B|$ and $|B| \leq |A|$, i.e., $|A| = |B|$. So any surjective homomorphism from $\boldsymbol{A}$ onto $\boldsymbol{B}$ must be also injective by the pigeonhole principle. Thus $\boldsymbol{A} \cong \boldsymbol{B}$.

$\cong$ is an equivalence relation on $\mathsf{Alg}(\Sigma)$. ($\Delta_A\colon \boldsymbol{A} \cong \boldsymbol{A}$; if $h\colon \boldsymbol{A} \cong \boldsymbol{B}$ and $g\colon \boldsymbol{B} \cong \boldsymbol{C}$, then $g \circ h\colon \boldsymbol{A} \cong \boldsymbol{C}$; if $h\colon \boldsymbol{A} \cong \boldsymbol{B}$ then $h^{-1}\colon \boldsymbol{B} \cong \boldsymbol{A}$.) The equivalence class of $[\boldsymbol{A}]_{\cong}$ of $\boldsymbol{A}$ under $\cong$, which we normally write simply as $[\boldsymbol{A}]$, is called the *isomorphism type* of $\boldsymbol{A}$. ($[\boldsymbol{A}]$ is not a proper set, it's too big, but this problem can be disregarded for our purposes.) The class of all isomorphism types of Σ-algebras is denoted by $[\mathsf{Alg}(\Sigma)]$.

The relations of subalgebra and homomorphic image on $\mathsf{Alg}(\Sigma)$ induce corresponding relations on $[\mathsf{Alg}(\Sigma)]$.

- $[\boldsymbol{A}] \subseteq [\boldsymbol{B}]$ if $\boldsymbol{A} \cong ; \subseteq \boldsymbol{B}$, i.e., if $\exists \boldsymbol{C}(\boldsymbol{A} \cong \boldsymbol{C} \subseteq \boldsymbol{B})$.
- $[\boldsymbol{A} \preccurlyeq \boldsymbol{B}]$ if $\boldsymbol{A} \preccurlyeq \boldsymbol{B}$. (Note that because $\cong \subseteq \preccurlyeq$, $(\cong ; \preccurlyeq) \subseteq (\preccurlyeq ; \preccurlyeq) = \preccurlyeq$.)

$\subseteq$ and $\preccurlyeq$ are well defined in isomorphism types, i.e., if $\boldsymbol{A} \cong \boldsymbol{A}'$ and $\boldsymbol{B} \cong \boldsymbol{B}'$, then $\boldsymbol{A} \cong ; \subseteq \boldsymbol{B}$ iff $\boldsymbol{A}' \cong ; \subseteq \boldsymbol{B}'$ and $\boldsymbol{A} \preccurlyeq \boldsymbol{B}$ iff $\boldsymbol{A}' \preccurlyeq \boldsymbol{B}'$.

To see that these equivalences holds we observe that $\boldsymbol{A} \cong ; \subseteq \boldsymbol{B}$ implies $\boldsymbol{A}' \cong ; \cong ; \subseteq ; \cong \boldsymbol{B}'$, and $\boldsymbol{A} \preccurlyeq \boldsymbol{B}$ implies $\boldsymbol{A} \cong ; \preccurlyeq ; \cong \boldsymbol{B}$. The second implication holds because clearly $\cong ; \preccurlyeq \, = \, \preccurlyeq ; \cong \, = \, \preccurlyeq$. The first implication is an immediate consequence of the equality $\subseteq ; \cong \, = \, \cong ; \subseteq$, which is in turn a corollary of Thm. 2.15(i) below.

$\subseteq$ is a partial ordering of isomorphism types, and $\preccurlyeq$ is what is called a *quasi-ordering* or *pre-ordering*, i.e., it is reflexive and transitive but not symmetric. However, $\preccurlyeq$ is a partial ordering on *finite isomorphism types*, that is isomorphism types of finite algebras. Clearly, if $[\boldsymbol{A}] \preccurlyeq [\boldsymbol{B}]$ and $[\boldsymbol{B}] \preccurlyeq [\boldsymbol{C}]$ and $\boldsymbol{A}$ (equivalently $\boldsymbol{B}$) is finite, then $[\boldsymbol{A}] = [\boldsymbol{B}]$.

Let us consider the various relative products of $\subseteq$ and $\preccurlyeq$ and their converses:

$$\subseteq ; \preccurlyeq, \quad \preccurlyeq ; \subseteq, \quad \subseteq ; \succcurlyeq, \quad \succcurlyeq ; \subseteq .$$

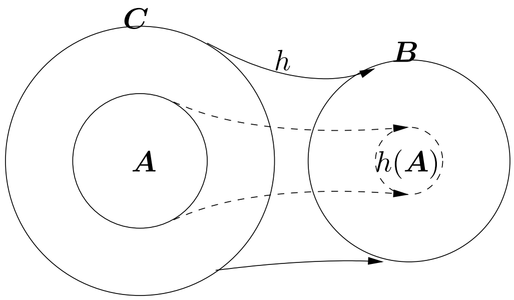
This gives half of the eight possible combinations, but each of the remaining four is a converse of one of these. For example, $(\supseteq ; \preccurlyeq) = (\breve{\subseteq} ; \breve{\succcurlyeq}) = (\subseteq ; \preccurlyeq)^{\smile}$.

Theorem 2.15. *The following inclusions as relations be Σ-isomorphism types.*

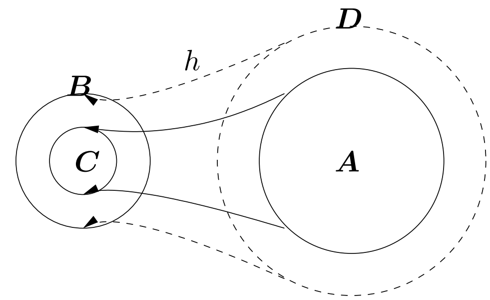
(i) $\subseteq ; \succcurlyeq \, = \, \succcurlyeq ; \subseteq$.
(ii) $\subseteq ; \preccurlyeq \, \subset \, \preccurlyeq ; \subseteq$.

Proof. (i) $\subseteq$. Assume $[\boldsymbol{A}] \subseteq ; \succcurlyeq [\boldsymbol{B}]$, i.e., there exists a $\boldsymbol{C}$ such that $\boldsymbol{A} \subseteq \boldsymbol{C} \succcurlyeq \boldsymbol{B}$. We need to show $[\boldsymbol{A}] \succcurlyeq ; \subseteq [\boldsymbol{B}]$, i.e., there exists a $\boldsymbol{D}$ such that $\boldsymbol{A} \succcurlyeq \boldsymbol{D} \subseteq \boldsymbol{B}$. Let $h\colon \boldsymbol{C} \twoheadrightarrow \boldsymbol{B}$. Then $\boldsymbol{A} \succcurlyeq h(\boldsymbol{A}) \subseteq \boldsymbol{B}$. See the following figure.

The inclusion $\supseteq$ of (i) is left as an exercise. See the following figure.



$\subseteq ; \succcurlyeq \subseteq \succcurlyeq ; \subseteq$



$\subseteq ; \succcurlyeq \supseteq \succcurlyeq ; \subseteq$

(ii) $\subseteq$. Assume $\boldsymbol{A} \subseteq \boldsymbol{C} \preccurlyeq \boldsymbol{B}$. Let $h\colon \boldsymbol{B} \twoheadrightarrow \boldsymbol{C}$. Then $\boldsymbol{A} \preccurlyeq h^{-1}(\boldsymbol{A}) \subseteq \boldsymbol{B}$.

We show by example that the inclusion of (ii) is proper. Let $\mathbb{Q} = \langle \mathbb{Q}, +, \cdot, -, 0, 1\rangle$ be the ring of rational numbers (a field). Recall that $\mathbb{Z} = \langle \mathbb{Z}, +, \cdot, -, 0, 1\rangle$ is the ring of integers and let $\mathbb{Z}_2 = \langle \mathbb{Z}_2, +, \cdot, -, 0, 1\rangle$ be the ring of integers $\pmod 2$. We know that $\mathbb{Z}_2 \preccurlyeq \mathbb{Z} \subseteq \mathbb{Q}$, so $\mathbb{Z}_2 \preccurlyeq ; \subseteq \mathbb{Q}$. But it is not the case that $\mathbb{Z}_2 \subseteq ; \preccurlyeq \mathbb{Q}$. In fact, we show that

$$\mathsf{H}(\{\mathbb{Q}\}) = \mathsf{I}(\{\mathbb{Q}\}) \cup \{\, \boldsymbol{A} \in \mathsf{Alg}(\Sigma) : |A| = 1 \,\}, \tag{21}$$

i.e., the only nontrivial (two or more elements) homomorphic images of $\mathbb{Q}$ are its isomorphic images. Suppose $h\colon \mathbb{Q} \twoheadrightarrow \boldsymbol{A}$, and suppose h is not an isomorphism, i.e., it is not injective. Let a and b be distinct elements of $\mathbb{Q}$ such that $h(a) = h(b)$. $a - b \neq 0$ but $h(a-b) = h(a + -b) = h(a) + -h(b) = h(a) - h(b) = 0$. Thus $1 = h(1) = h((a-b)\cdot(a-b^{-1})) = h(a-b)\cdot h((a-b)^{-1}) = 0 \cdot h((a-b)^{-1}) = 0$. So for every $a \in A$, $a = 1\cdot a = 0 \cdot a = 0$; i.e., $\boldsymbol{A}$ is trivial. This proves the claim.

Suppose now by way of contradiction that for some $\boldsymbol{A}$, $\mathbb{Z}_2 \subseteq \boldsymbol{A} \preccurlyeq \mathbb{Q}$. By the claim $\boldsymbol{A}$ must be either isomorphic to $\mathbb{Q}$ or a trivial one-element algebra. But $\mathbb{Z}_2$ is not isomorphic to an subalgebra of $\mathbb{Q}$. □

For any class K of Σ-algebras, we define

$$\mathsf{H}(\mathsf{K}) = \{\, \boldsymbol{A} \in \mathsf{Alg}(\Sigma) : \exists \boldsymbol{B} \in \mathsf{K}(\boldsymbol{A} \preccurlyeq \boldsymbol{B}) \,\},$$
$$\mathsf{I}(\mathsf{K}) = \{\, \boldsymbol{A} \in \mathsf{Alg}(\Sigma) : \exists \boldsymbol{B} \in \mathsf{K}(\boldsymbol{A} \cong \boldsymbol{B}) \,\},$$

the classes respectively of homomorphic and isomorphic images of algebras of K. H and I are algebraic closure operators on $\mathsf{Alg}(\Sigma)$. For example $\mathsf{H}\,\mathsf{H}(\mathsf{K}) = \mathsf{H}(\mathsf{K})$ because of the transitivity of $\preccurlyeq$. H is algebraic because $\mathsf{H}(\mathsf{K}) = \bigcup\{\, \mathsf{H}(\boldsymbol{A}) : \boldsymbol{A} \in \mathsf{K} \,\}$.

Theorem 2.16. *For any class* K *of* Σ*-algebras,*

(i) $\mathsf{S}\,\mathsf{H}(\mathsf{K}) \subseteq \mathsf{H}\,\mathsf{S}(\mathsf{K})$.
(ii) $\mathsf{H}\,\mathsf{S}$ *is an algebraic closure operator on* $\mathsf{Alg}(\Sigma)$.

Proof. (i). Suppose $\boldsymbol{A} \in \mathsf{S}\,\mathsf{H}(\mathsf{K})$. Then there exists a $\boldsymbol{B} \in \mathsf{K}$ such that $\boldsymbol{A} \subseteq ; \preccurlyeq \boldsymbol{B}$. Then by Thm. 2.15(ii), $\boldsymbol{A} \preccurlyeq ; \subseteq \boldsymbol{B}$. Thus $\boldsymbol{A} \in \mathsf{H}\,\mathsf{S}(\mathsf{K})$.

(ii) $\mathsf{K} \subseteq \mathsf{S}(\mathsf{K})$ by the extensivity of K, and hence by the extensivity and monotonicity of H, $\mathsf{K} \subseteq \mathsf{H}(\mathsf{K}) \subseteq \mathsf{H}\,\mathsf{S}(\mathsf{K})$. So $\mathsf{H}\,\mathsf{S}$ is extensive. $\mathsf{H}\underline{\mathsf{S}\,\mathsf{H}}\mathsf{S}(\mathsf{K}) \overset{(i)}{\subseteq} \mathsf{H}\underline{\mathsf{H}\,\mathsf{S}}\mathsf{S}(\mathsf{K}) = \mathsf{H}\,\mathsf{S}(\mathsf{K})$. Since clearly $\mathsf{H}\,\mathsf{S}\,\mathsf{H}\,\mathsf{S}(\mathsf{K}) \subseteq \mathsf{H}\,\mathsf{S}(\mathsf{K})$, we get that $\mathsf{H}\,\mathsf{S}$ is idempotent. Finally, $\mathsf{K} \subseteq \mathsf{L}$ implies $\mathsf{S}(\mathsf{K}) \subseteq \mathsf{S}(\mathsf{L})$ which in turn implies $\mathsf{H}\,\mathsf{S}(\mathsf{K}) \subseteq \mathsf{H}\,\mathsf{S}(\mathsf{L})$. So $\mathsf{H}\,\mathsf{S}$ is monotonic. $\boldsymbol{A} \in \mathsf{H}\,\mathsf{S}(\mathsf{K})$ iff there is a $\boldsymbol{B} \in \mathsf{K}$ such that $\boldsymbol{A} \preccurlyeq ; \subseteq \boldsymbol{B}$. Thus $\mathbf{HS}(\mathsf{K}) \subseteq \bigcup\{\, \mathbf{HS}(\boldsymbol{B}) : \boldsymbol{B} \in \mathsf{K}\,\}$. Thus $\mathbf{HS}$ is algebraic. □

From Thm. 2.15(i) we see that the opposite inclusion of Thm. 2.16(i) does not in hold in general.

We also note the following obvious identities, which prove useful later. $\mathbf{I}(\mathsf{K}) = \mathbf{HI}(\mathsf{K}) = \mathbf{H}(\mathsf{K})$ and $\mathbf{IS}(\mathsf{K}) = \mathbf{SI}(\mathsf{K})$.

Let Σ be a multi-sorted signature with sort set S. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be Σ-algebras. A *homomorphism* $h\colon \boldsymbol{A} \to \boldsymbol{B}$ is a S*-sorted map* $h = \langle h_s : s \in S \rangle$ such that, for all $s \in S$, $h_s\colon A_s \to B_s$, and for all $\sigma \in \Sigma$ of type $s_1, \dots, s_n \to s$ and for all $\langle a_1, \dots, a_n \rangle \in A_{s_1} \times \cdots \times A_{s_n}$, $h_s\big(\sigma^{\boldsymbol{A}}(a_1, \dots, a_n)\big) = \sigma^{\boldsymbol{B}}\big(h_{s_1}(a_1), \dots, h_{s_n}(a_n)\big)$.

Example. Let A and B be nonempty sets. Recall that

$$\mathbf{Lists}(A) = \big\langle \langle A \cup \{e_D\}, A^* \cup \{e_l\} \rangle, \texttt{head}, \texttt{tail}, \texttt{append}, \texttt{emptylist}, \texttt{derror}, \texttt{derror} \big\rangle.$$

Let $f\colon A \to B$ be any map. We define the S-sorted map $h = \langle h_D, h_L \rangle$ where $h_D{\restriction}A = f$ and $h_D(e_D) = e_D$, and, for all $a_1, \dots, a_n \in D$, $h_L(\langle a_1, \dots, a_n \rangle) = \langle f(a_1), \cdots, f(a_n) \rangle$, and $h_L(e_l) = e_l$. Then $h \in \mathrm{Hom}(\boldsymbol{A}, \boldsymbol{B})$ and every homomorphism from $\boldsymbol{A}$ to $\boldsymbol{B}$ comes from some $f\colon A \to B$ in this way (Exercise).

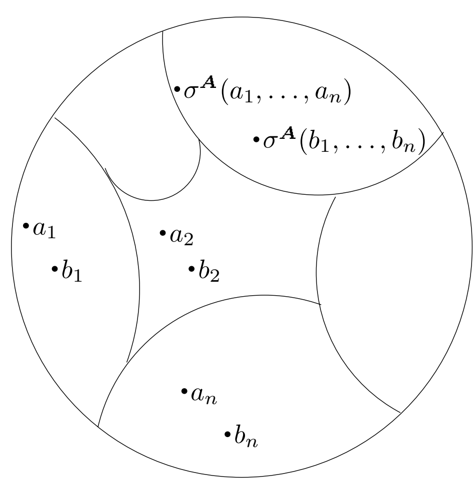
2.5. **Congruence relations and quotient algebras.**

Definition 2.17. Let $\boldsymbol{A}$ be a Σ-algebra. An equivalence relation E on A is called a *congruence relation* if, for all $n \in \boldsymbol{\omega}$, all $\sigma \in \Sigma_n$, and all $a_1, \dots, a_n, b_1, \dots, b_n \in A$,

$$a_1 \mathrel{E} b_1, \dots, a_n \mathrel{E} b_n \text{ imply } \sigma^{\boldsymbol{A}}(a_1, \dots, a_n) \mathrel{E} \sigma^{\boldsymbol{A}}(b_1, \dots, b_n). \tag{22}$$

The set of all congruences on $\boldsymbol{A}$ is denoted by $\mathrm{Co}(\boldsymbol{A})$.

(22) is called the *substitution property*. Intuitively, it asserts that the equivalence class of the result of applying any one of the fundamental operations of $\boldsymbol{A}$ depends only on the equivalence classes of the arguments. See the following figure.



We use lower case Greek letters, e.g., α, β, γ, etc., to represent congruence letters. The equivalence class $[a]_\alpha$ of a is called the *congruence class* of a and is normally denoted by

a/α. So $\alpha\,\alpha\,b$ iff $a/\alpha = b/\alpha$ iff $a \in b/\alpha$. The set of all congruence classes of α, i.e., the partition of α, is denoted by A/α.

Definition 2.18. Let $\boldsymbol{A}$ be a Σ-algebra and let $\alpha \in \mathrm{Co}(\boldsymbol{A})$. We define an Σ-algebra

$$\boldsymbol{A}/\alpha = \langle A/\alpha,\, \sigma^{\boldsymbol{A}/\alpha}\rangle_{\sigma\in\Sigma}$$

on the set of congruence classes of α as follows. For every $n \in \boldsymbol{\omega}$, every $\sigma \in \Sigma_n$, and for all $a_1/\alpha, \dots, a_n/\alpha \in A/\alpha$,

$$\sigma^{\boldsymbol{A}/\alpha}(a_1/\alpha, \dots, a_n/\alpha) = \sigma^{\boldsymbol{A}}(a_1, \dots, a_n)/\alpha.$$

$\boldsymbol{A}/ga$ is called a *quotient algebra*, or more precisely, the *quotient of* $\boldsymbol{A}$ *by* α.

Note that $\sigma^{\boldsymbol{A}/\alpha}$ is well defined by the substitution property.

Examples. (1) $\Delta_A, \nabla_A \in \mathrm{Co}(\boldsymbol{A})$. $\boldsymbol{A}/\Delta_A \cong \boldsymbol{A}$ and $\boldsymbol{A}/\nabla_A$ is a trivial one-element algebra.

(2) Let $\boldsymbol{G} = \langle G, \cdot, ^{-1}, e\rangle$ be a group. A subuniverse N of $\boldsymbol{G}$ is called *normal* if it is closed under *conjugation* by arbitrary elements of G, i.e., if $a \in N$ implies $x \cdot a \cdot x^{-1} \in N$ for every $x \in G$. Define $a\,\alpha\,b$ if $a \cdot b^{-1} \in N$. Then α is a congruence relation. Furthermore, for each congruence α on $\boldsymbol{G}$, e/α is a normal subuniverse of Γ. The mapping $\alpha \mapsto e/\alpha$ is a bijection between $\mathrm{Co}(\boldsymbol{G})$ and the normal subuniverses of $\boldsymbol{G}$. We leave the verification of all these facts to the reader but will give the details of a similar verification for rings.

(3) Let $\boldsymbol{R} = \langle R, +, \cdot, -, 0\rangle$ be a ring, and let I be an *ideal* subuniverse of $\boldsymbol{R}$: if $a \in I$ then $x \cdot a, a \cdot x \in I$ for every $x \in R$. Define $a\,\alpha\,b$ if $a - b$ $(= a + -b)$ is in I. We show α is an equivalence relation. $a - a = 0 \in I$. $a - b, b - c \in I$ imply $a - c = (a-b)+(b-c) \in I$. $a - b \in I$ implies $b - a = -(a-b) \in I$. We now verify the substitution property. Suppose $a_1\,\alpha\,b_1$ and $a_2\,\alpha\,b_2$, i.e., $a_1 - b_1, a_2 - b_2 \in I$. Then $(a_1 + a_2) - (b_1 + b_2) = (a_1 - b_1) + (a_2 - b_2) \in I$. So $(a_1+a_2)\,\alpha\,(b_1+b_2)$. $-a_1 - (-b_1) = -(a_1 - b_1) \in I$. So $-a_1\,\alpha\,-b_1$. Finally, $\big((a_1-b_1)\cdot a_2\big)+\big(b_1\cdot(a_2-b_2)\big) \in I$. But $\big((a_1-b_1)\cdot a_2\big)+\big(b_1\cdot(a_2-b_2)\big) = a_1\cdot a_2 - b_1\cdot a_2 + b_1\cdot a_2 - b_1\cdot b_2 = a_1\cdot a_2 - b_1\cdot b_2$. So $(a_1 \cdot a_2)\,\alpha\,(b_1 \cdot b_2)$. Thus α is a congruence on $\boldsymbol{G}$.

Now let $\alpha \in \mathrm{Co}(\boldsymbol{G})$. We check that $0/\alpha$ is an ideal. Let $a, a' \in 0/\alpha$, i.e., $a\,\alpha\,a'\,\alpha\,0$. Then $(a + a')\,\alpha\,(0 + 0) = 0$, $-a\ \alpha\ -0 = 0$, $(a \cdot a')\,\alpha\,(0 \cdot 0) = 0$. So $0/\alpha$ is a subuniverse. Moreover, for every $x \in R$, $(x \cdot a)\,\alpha\,(x \cdot 0) = 0$ and $(a \cdot x)\,\alpha\,(0 \cdot x) = 0$. So $0/\alpha$ is an ideal. Furthermore, $(a - b) \in 0/\alpha$ iff $(a - b)\,\alpha\,0$ iff $a = ((a - b) + b)\,\alpha\,(0 + b) = b$. So α is the congruence determined as in the first part by the ideal $0/\alpha$. Conversely, if one starts with an ideal I, constructs the congruence α as in the first part, then forms its ideal $0/\alpha$ one gets back I, because $a \in 0/\alpha$ iff $a\,\alpha\,0$ iff $a = (a - 0) \in I$. So for any ring $\boldsymbol{R}$ the mapping $\alpha \to 0/\alpha$ is a bijection between $\mathrm{Co}(\boldsymbol{R})$ and the set of ideals of $\boldsymbol{R}$.