2.2. **The structure of mono-unary algebras.** Let $\boldsymbol{A} = \langle A, f\rangle$, where $f\colon A \to A$. By a *finite chain* in $\boldsymbol{A}$ we mean either a subset of A of the form $\{a, f^1(a), f^2(a), \dots, f^n(a)\}$, for some $n \in \omega$, or of the form $\{a, f^1(a), f^2(a), \dots\} = \{\, f^k(a) : k \in \omega\,\}$, where $f^i(a) \neq f^j(a)$ for all $i < j \leq n$ in the first case and for all $i < j < \omega$ in the second. Note that the finite chain is not a subuniverse of $\boldsymbol{A}$ unless $f^{n+1}(a) = f^i(a)$ for some $i \leq n$. The infinite chain is clearly a subuniverse and is isomorphic to the natural numbers under successor; we call it an *ω-chain*. By a *cycle* we mean a finite subset of $\boldsymbol{A}$ of the form $\{a, f^1(a), f^2(a), \dots, f^{p-1}(a)\}$, where $f^p(a) = a$ but $f^i(a) \neq f^j(a)$ for all $i < j < p$. p is called the *period* the the cycle. A cycle is clearly a subuniverse.

We first consider the case where $\boldsymbol{A}$ is *cyclic*, i.e., $A = \mathrm{Sg}^{\boldsymbol{A}}(\{a\})$. It is easy to see that $A = \{\, f^n(a) : n \in \omega\,\}$. We show that if $\boldsymbol{A}$ is finite, then it must be in the form of a finite chain that is attached at the end to a cycle; see Figure 1.

Suppose $f^n(a) = f^m(a)$ for some $n < m$. Let l be the least n such that there exists an $m > n$ such that $f^n(a) = f^m(a)$. Then let p be the least $k > 0$ such that $f^{l+k}(a) = f^l(a)$. p is called the *period* of $\boldsymbol{A}$ (and of a) and l is called its *tail length*. $\boldsymbol{A}$ is finite. $\boldsymbol{A}$ thus consists of a finite chain of length l, called the *tail* of $\boldsymbol{A}$, that is attached to a cycle of period p, called the *cycle* of $\boldsymbol{A}$. If $f^n(a) \neq f^m(a)$ for all distinct $n, m \in \omega$, then $\boldsymbol{A}$ is an infinite

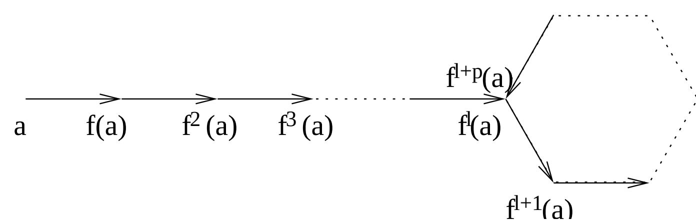


Figure 1

ω-chain.

Elements a, b of an arbitrary mono-unary algebra $\boldsymbol{A}$ are said to be *connected* if there exist $n, m \in \omega$ such that $f^n(a) = f^m(b)$. The relation of being connected is an equivalence relation. It is clearly reflexive and symmetric. Suppose $f^n(a) = f^m(b)$ and $f^k(b) = f^l(c)$. Then $f^{n+k}(a) = f^{m+k}(b) = f^{l+m}(c)$. The equivalence classes are called *connected components* of $\boldsymbol{A}$. Each connected component C of $\boldsymbol{A}$ is a subuniverse. For if $a \in C$, then $f(f(a) = f^2(a)$; hence $f(a)$ is connected to a and thus in C. $\boldsymbol{A}$ is the disjoint union of its connected components, and hence in order to fully understand the structure of mono-unary algebras it suffices to focus on connected algebras (those with a single connected component) Clearly any cyclic algebra is connected.

We now consider the proper 2-generated connected algebras, i.e., $A = \mathrm{Sg}^{\boldsymbol{A}}(\{a, b\})$ and $\boldsymbol{A}$ is not cyclic but is connected, i.e., there exist $n, m \in \omega$ such that $f^n(a) \neq f^m(b)$ but $f^{n+1}(a) = f^{m+1}(b)$. Since they are connected, $\mathrm{Sg}^{\boldsymbol{A}}(\{a\})$ is finite iff $\mathrm{Sg}^{\boldsymbol{A}}(\{b\})$ is, and in this case they have the same cycle. The tails either attach separately to the cycle or merge before the cycle, see Figure 2. $\boldsymbol{A}$ is infinite iff $\mathrm{Sg}^{\boldsymbol{A}}(\{a\})$ and $\mathrm{Sg}^{\boldsymbol{A}}(\{b\})$ are both infinite. It can be viewed either as the ω-chain $\mathrm{Sg}^{\boldsymbol{A}}(\{b\})$ with a finite chain beginning with b attached, or as the ω-chain $\mathrm{Sg}^{\boldsymbol{A}}(\{b\})$ with a finite chain beginning at a attached; see Figure 2

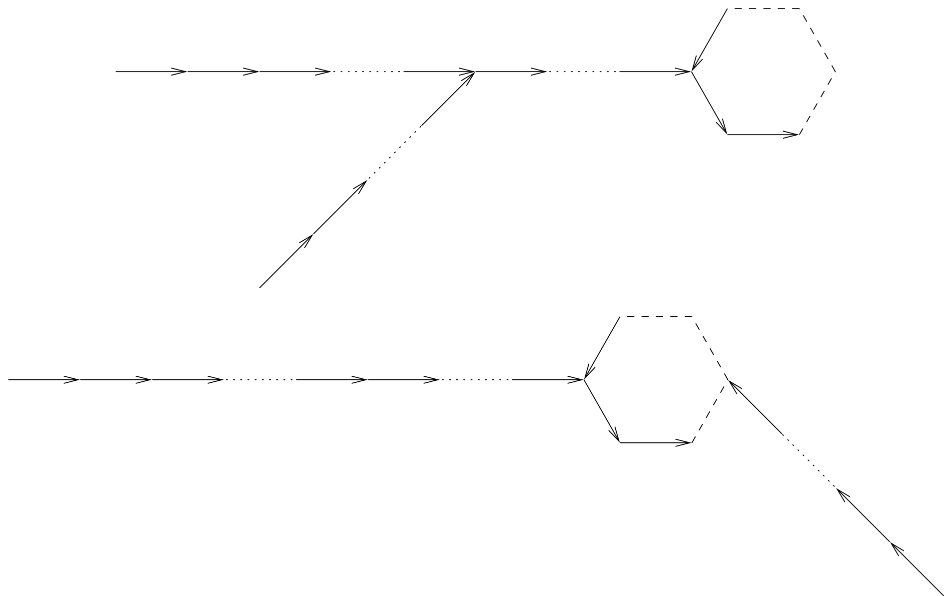


FIGURE 2

The proper 3-generated connected algebras are of the following form: three finite algebras, one with the three tails separately attached to the cycle; one with two of the tails merging before the cycle; and one with all three tails merging before the cycle. The one infinite form is an ω-chain with two finite chains attached to it. By a *finite reverse tree* we mean mean a finite chain with a finite number of finite chains attached to it. Every finite, finitely generated, connected mono-unary algebra is a cycle with a finite number of finite reverse trees attached to it. Every infinite, finitely generated, connected mono-unary algebra is an ω-chain with a finite number of reverse trees attached to it.

Examples of a nonfinitely generate mono-unary connected algebras are the natural numbers under the predecessor (a *reverse ω-chain*) attached to a cycle, and a ω-chain and a reverse ω-chain put together, i.e., the integers under successor. A full description of the nonfinitely generated mono-unary connected algebras is left to the reader.

2.3. **Subalgebras.** Roughly speaking a subalgebra of an algebra is a nonempty subuniverse with together with the algebraic structure it inherits from its parent.

Definition 2.8. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be Σ-algebras. $\boldsymbol{B}$ is a *subalgebra* of $\boldsymbol{A}$, in symbols $\boldsymbol{B} \subseteq \boldsymbol{A}$, if $B \subseteq A$ and, for every $\sigma \in \Sigma$ and all $b_1, \dots, b_n \in B$ (n is the rank of σ), $\sigma^{\boldsymbol{B}}(b_1, \dots, b_n) = \sigma^{\boldsymbol{A}}(b_1, \dots, b_n)$.

If $\boldsymbol{B} \subseteq \boldsymbol{A}$, then $B \in \mathrm{Sub}(\boldsymbol{A})$. Conversely, if $B \in \mathrm{Sub}(\boldsymbol{A})$ and $B \neq \emptyset$, then there is a unique $\boldsymbol{B} \subseteq \boldsymbol{A}$ such that B is the universe of $\boldsymbol{B}$.

Let $\mathsf{Alg}(\Sigma)$ be the class of all Σ-algebras. $\subseteq$ is a partial ordering of $\mathsf{Alg}(\Sigma)$. It is clearly reflexive and antisymmetric. If $\boldsymbol{C} \subseteq \boldsymbol{B}$ and $\boldsymbol{B} \subseteq \boldsymbol{A}$, then $C \subseteq B$ and $B \subseteq A$ so $C \subseteq A$, and for all $c_1, \dots, c_n \in C$, $\sigma^{\boldsymbol{C}}(c_1, \dots, c_n) = \sigma^{\boldsymbol{B}}(c_1, \dots, c_n) = \sigma^{\boldsymbol{A}}(c_1, \dots, c_n)$. $\langle \mathsf{Alg}(\Sigma), \subseteq \rangle$ is not a lattice ordering. If $A \cap B = \emptyset$, then $\boldsymbol{A}$ and $\boldsymbol{B}$ cannot have a GLB. Allowing empty algebras would clearly not alter the situation for signatures with constants, and it is not hard to see that the the same is true even for signatures without constants. The problem becomes more interesting when we consider *isomorphism types* of algebras below.

For any class K of Σ-algebras we define

$$\mathbf{S}(\mathsf{K}) = \{\, \boldsymbol{A} \in \mathsf{Alg}(\Sigma) : \text{ there exists a } \boldsymbol{B} \in \mathsf{K} \text{ such that } \boldsymbol{A} \subseteq \boldsymbol{B} \,\}.$$

For simplicity we write $\mathbf{S}(\boldsymbol{A})$ for $\mathbf{S}(\{\boldsymbol{A}\})$.

$\mathbf{S}$ is an algebraic closure operator on $\mathsf{Alg}(\Sigma)$. Clearly $\mathsf{K} \subseteq \mathbf{S}(\mathsf{K})$ by the reflexivity of $\subseteq$, and and $\mathbf{S}\,\mathbf{S}(\mathsf{K}) = \mathbf{S}(K)$ because $\subseteq$ is transitive. Also $\mathsf{K} \subseteq \mathsf{L}$ implies $\mathbf{S}(\mathsf{K}) \subseteq \mathbf{S}(\mathsf{L})$. And $\mathbf{S}(\mathsf{K}) = \bigcup\{\, \mathbf{S}(\mathsf{K}') : \mathsf{K}' \subseteq_\omega \mathsf{K} \,\}$. In fact, $\mathbf{S}(\mathsf{K}) = \bigcup\{\, \mathbf{S}(\boldsymbol{A}) : \boldsymbol{A} \in \mathsf{K} \,\}$.

We should mention here that there are some set-theoretical difficulties in dealing with the class of all Σ-algebras because it is too large. Technically it is a *proper class* and not a set; a set can be an element of a class but a class cannot. Thus although the class $\mathsf{Alg}(\Sigma)$ of all algebras of signature Σ exists, the class $\{\mathsf{Alg}(\Sigma)\}$ whose only member is $\mathsf{Alg}(\Sigma)$ does not. In the sequel for purposes of simplicity and convenience we will use notation and terminology that in their normal set-theoretical interpretation implies that we are assuming the existence of classes that contain $\mathsf{Alg}(\Sigma)$ as an element. But the actual interpretation makes no assumption of this kind and is consistent with standard set-theory.

2.4. **Homomorphisms and quotient algebras.** Let $h: A \to B$ be a mapping between the sets A and B. h is *surjective* or *onto* if the range and codomain of h are the same, i.e., $h(A) = B$; we write $h: A \twoheadrightarrow B$ in this case. h is *injective* or *one-one* if, for all $a, a' \in A$, $a \neq a'$ implies $h(a) \neq h(a')$; we write $h: A \rightarrowtail B$. Finally, h is *bijective* or *one-one onto* if it is both surjective and injective, in symbols, $h: A \cong B$.

Definition 2.9. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be Σ-algebras. A mapping $h: A \to B$ is a *homomorphism*, in symbols $h: \boldsymbol{A} \to \boldsymbol{B}$, if, for all $\sigma \in \Sigma$ and all $a_1, \dots, a_n \in A$, with $n = \rho(\sigma)$,

$$h\big(\sigma^{\boldsymbol{A}}(a_1, \dots, a_n)\big) = \sigma^{\boldsymbol{B}}(h(a_1), \dots, h(a_n)).$$

A surjective homomorphism is called an *epimorphism* ($h: \boldsymbol{A} \twoheadrightarrow \boldsymbol{B}$) and an injective homomorphism is called a *monomorphism* ($h: \boldsymbol{A} \rightarrowtail \boldsymbol{B}$). A bijective homomorphism is called an isomorphism and is written either $h: \boldsymbol{A} \cong \boldsymbol{B}$.

A homomorphism with the same domain and codomain, i.e., $h: \boldsymbol{A} \to \boldsymbol{A}$, is called and *endomorphism* of $\boldsymbol{A}$, and an isomorphism with the same domain and codomain, i.e., $h: \boldsymbol{A} \cong \boldsymbol{A}$, is an *automorphism* of $\boldsymbol{A}$.

$\mathrm{Hom}(\boldsymbol{A}, \boldsymbol{B})$ will denote the set of all homomorphisms from $\boldsymbol{A}$ to $\boldsymbol{B}$. $\mathrm{Iso}(\boldsymbol{A}, \boldsymbol{B})$, $\mathrm{End}(\boldsymbol{A})$, and $\mathrm{Aut}(\boldsymbol{A})$ are defined accordingly.

Examples. The classic example is the homomorphism from the additive group of integers $\boldsymbol{\mathbb{Z}} = \langle \mathbb{Z}, +, -, 0 \rangle$ to the group of integers $\pmod n$ $\boldsymbol{\mathbb{Z}}_n = \langle \mathbb{Z}_n, + \pmod n, - \pmod n, 0 \pmod n \rangle$.

For $n \in \mathbb{Z}$, the mapping $h_n: \mathbb{Z} \to \mathbb{Z}$ defined by

$$h_n(x) = nx = \begin{cases} \underbrace{x + \cdots + x}_{n} & \text{if } n > 0, \\ 0 & \text{if } n = 0, \\ \underbrace{-x + - \cdots + -x}_{-n} & \text{if } n < 0 \end{cases}$$

is an endomorphism of $\mathbf{Z}$: $h(x+y) = n(x+y) = nx + ny$; $h(-x) = n(-x) = -(nx)$; $h(0) = n0 = 0$.

Theorem 2.10. *Let* $\boldsymbol{A} = \langle A, \cdot, ^{-1}, e\rangle$ *and* $\boldsymbol{B} = \langle B, \cdot, ^{-1}, e\rangle$ *be groups. Then* $\mathrm{Hom}(\boldsymbol{A},\boldsymbol{B}) = \mathrm{Hom}(\langle A, \cdot\rangle, \langle B, \cdot\rangle)$.

Proof. Clearly $\mathrm{Hom}(\boldsymbol{A},\boldsymbol{B}) \subseteq \mathrm{Hom}(\langle A, \cdot\rangle, \langle B, \cdot\rangle)$. Let $h \in \mathrm{Hom}(\langle A, \cdot\rangle, \langle B, \cdot\rangle)$. $h(e)\cdot h(e) = h(e \cdot e) = h(e) = e \cdot h(e)$. So $h(e) = e$ by cancellation. $h(a^{-1}) \cdot h(a) = h(a^{-1} \cdot a) = e = h(a)^{-1} \cdot h(a)$. So $h(a^{-1}) = h(a)^{-1}$ by cancellation. □

$\langle \mathbb{Z}, +\rangle$ is called a *reduct* of $\mathbf{Z}$. There is a useful general notion of reduct. Let $\langle \Sigma, \rho_\Sigma\rangle$ and $\langle \Delta, \rho_\Sigma\rangle$ be signatures. Δ is a *subsignature* of Σ if $\Delta \subseteq \Sigma$ and, for each $\delta \in \Delta$, $\rho_\Delta(\delta) = \rho_\Sigma(\delta)$.

Definition 2.11. Let Σ be a signature and $\boldsymbol{A}$ a Σ-algebra. Then for every subsignature Δ of Σ, the Δ-algebra $\langle A, \delta^{\boldsymbol{A}}\rangle_{\delta\in\Delta}$ is called the Δ-*reduct* of $\boldsymbol{A}$. It is denoted by $\mathrm{Red}_\Delta(\boldsymbol{A})$.

Clearly, for all Σ-algebras $\boldsymbol{A}$ and $\boldsymbol{B}$ and every subsignature Δ of Σ, $\mathrm{Sub}(\boldsymbol{A}) \subseteq \mathrm{Sub}(\mathrm{Red}_\Delta(\boldsymbol{A}))$. We have seen that the equality fails to holds for the $\{+\}$-reduct of $\mathbf{Z}$. It does hold however for the $\{\cdot\}$-reduct of any finite group (exercise). Exercise: Is it true in general that $\mathbf{Sub}(\boldsymbol{A})$ is a sublattice of $\mathbf{Sub}(\mathrm{Red}_\Delta(\boldsymbol{A}))$?

It is also clear that $\mathrm{Hom}(\boldsymbol{A},\boldsymbol{B}) \subseteq \mathrm{Hom}(\mathrm{Red}_\Delta(\boldsymbol{A}), \mathrm{Red}_\Delta(\boldsymbol{B}))$, and we showed above that equality holds for the $\{\cdot\}$-reduct of any group (finite or infinite).

Every endomorphism of $\mathbf{Z}$ is of the form h_n for some $n \in \omega$. To see this consider any $g \in \mathrm{End}(\mathbf{Z})$, and let $n = g(1)$. If $x > 0$,

$$g(x) = g(\underbrace{1 + \cdots + 1}_{x}) = \underbrace{g(1) + \cdots + g(1)}_{x} = nx = h_n(x).$$

If $x = 0$, $g(x) = 0 = h_n(x)$, and if $x < 0$,

$$g(x) = g(\underbrace{-1 + \cdots + -1}_{-x}) = \underbrace{-g(1) + \cdots + -g(1)}_{-x} = (-n)(-x) = nx = h_n(x).$$

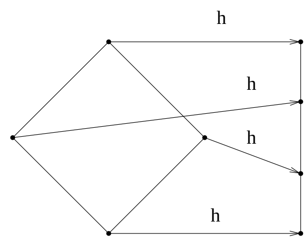
This result is a special case of a more general result which we now present.

Theorem 2.12. *Let* $\boldsymbol{A}, \boldsymbol{B}$ *be* Σ*-algebras, and assume* $\boldsymbol{A}$ *is a generated by* $X \subseteq A$*, i.e.,* $A = \mathrm{Sg}^{\boldsymbol{A}}(X)$*. Then every* $h \in \mathrm{Hom}(\boldsymbol{A},\boldsymbol{B})$ *is uniquely determined by its* restriction $h{\restriction}X$ *to* X*, i.e., for all* $h, h' \in \mathrm{Hom}(\boldsymbol{A},\boldsymbol{B})$*, if* $h{\restriction}X = h'{\restriction}X$*, then* $h = h'$.

Proof. The proof is by structural induction. Let $\mathcal{P}$ be the property of an element of $\boldsymbol{A}$ that its images under h and h' is the same; identifying a property with the set of all elements that have the property (this is called *extensionality*) we can say that $\mathcal{P} = \{\, a \in A : h(a) = h'(a)\,\}$. $X \subseteq \mathcal{P}$ by assumption. For every $\sigma \in \Sigma$ and all $a_1, \dots, a_n \in \mathcal{P}$, $h\big(\sigma^{\boldsymbol{A}}(a_1,\dots,a_n)\big) = \sigma^{\boldsymbol{B}}\big(h(a_1),\dots,h(a_n)\big) = \sigma^{\boldsymbol{B}}\big(h'(a_1),\dots,h'(a_n)\big) = h'\big(\sigma^{\boldsymbol{A}}(a_1,\dots,a_n)\big)$. So $\sigma^{\boldsymbol{A}}(a_1,\dots,a_n) \in \mathcal{P}$, and hence $\mathcal{P} \in \mathrm{Sub}(\boldsymbol{A})$. So $\mathcal{P} = A$ since X generates $\boldsymbol{A}$. □

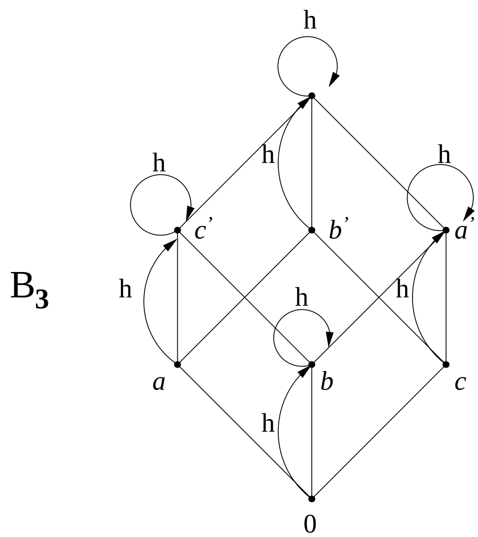
This theorem can be applied to give a easy proof that every endomorphism of $\mathbf{Z}$ is of the form h_n for some $n \in \mathbb{Z}$. Let $g \in \mathrm{End}(\mathbf{Z})$ and $n = g(1)$. Then $g(1) = h_n(1)$. Thus $g = h_n$ since $\mathbb{Z} = \mathrm{Sg}^{\mathbf{Z}}(\{1\})$.

Let $\boldsymbol{A} = \langle A, \vee, \wedge\rangle, \boldsymbol{B} = \langle B, \vee, \wedge\rangle$ be lattices. Then every $h \in \mathrm{Hom}(\boldsymbol{A},\boldsymbol{B})$ is order-preserving. In fact, $a \le a'$ implies $a \wedge a' = a'$ which in turn implies that $h(a) \vee h(a') = h(a')$, i.e., $h(a) \le h(a')$. The converse does not hold.



For example the mapping h in the figure above is order-preserving but is not a lattice homorphism. However recall that if if h is bijective and strictly order-preserving then it is a homorphism (Theorem 1.8).

Consider the function $h\colon B_3 \to B_3$ described pictorially in the following diagram, where $\boldsymbol{B}_3$ is the three-atom Boolean algebra, The claim is that h is endomorphism of $\boldsymbol{B}_3$. This



can be verified mechanically by considering each pair of elements x, y in turn and checking that $h(x \vee y) = h(x) \vee h(y)$ and $h(x \wedge y) = h(x) \wedge h(y)$, but this is a tedious process. For example, $h(c \vee b) = h(a') = a' = a' \vee b = h(a') \vee h(b)$. Here is a simpler way. Note first of all that, for all $x \in B_3$, $h(x) = x \vee b$. $\boldsymbol{B}_3$ is a distributive lattice. An easy way to see this is to observe that $\boldsymbol{B}_3$ is isomorphic to the $\langle \boldsymbol{\mathcal{P}}(\{1,2,3\}), \cup, \cap\rangle$, the lattice of all subsets of the three-element set $\{1, 2, 3\}$. The mapping $a \mapsto \{1\}$, $b \mapsto \{2\}$, $c \mapsto \{3\}$, $a' \mapsto \{2, 3\}$, $b' \mapsto \{1, 3\}$, $c' \mapsto \{1, 2\}$, $0 \mapsto \emptyset$, $1 \mapsto \{1, 2, 3\}$ is an order-preserving bijection and hence a lattice isomorphism.

So $\boldsymbol{B}_3$ is distributive. We use this fact to verify h is a homomorphism: $h(x \vee y) = (x \vee y) \vee b = (x \vee y) \vee (b \vee b) = (x \vee b) \vee (y \vee b) = h(x) \vee h(y)$, and $h(x \wedge y) = (x \wedge y) \vee b = (x \vee b) \wedge (y \vee b) = h(x) \wedge h(y)$.

Exercise: Prove that for every lattice $\boldsymbol{L}$ the mapping $x \mapsto x \vee a$ is an endomorphism of $\boldsymbol{L}$ for all $a \in A$ iff $\boldsymbol{L}$ is distributive.