A signature is *unary* if $\rho(\sigma) \leq 1$ for every $\sigma \in \Sigma$; *mono-unary* if $\Sigma = \{\sigma\}$ and $\rho(\sigma) = 1$; a *groupoid* if $\Sigma = \{\sigma\}$ and $\rho(\sigma) = 2$. An algebra is *unary*, *mono-unary*, a *groupoid* if its signature is. In the sequel, for each $n \in \omega$, $\Sigma_n = \{\, \sigma \in \Sigma : \rho(\sigma) = n \,\}$.

2.1. Subuniverses and subalgebras.

Definition 2.3. Let $\boldsymbol{A}$ be a Σ-algebra and $B \subseteq A$. B is a *subuniverse* of $\boldsymbol{A}$ if, for all $n \in \omega$, $\sigma \in \Sigma_n$, and $b_1, \dots, b_n \in B$, $\sigma^{\boldsymbol{a}}(b_1, \dots, b_n) \in B$, i.e., B is *closed* under $\sigma^{\boldsymbol{A}}$ for each $\sigma \in \Sigma$. The set of all subuniverses of $\boldsymbol{A}$ will be denoted by $\mathrm{Sub}(\boldsymbol{A})$.

Note that this implies $\sigma^{\boldsymbol{A}} \in B$ for every $\sigma \in \Sigma_0$, and that the empty set is a subuniverse of $\boldsymbol{A}$ iff $\Sigma_0 = \emptyset$, i.e., $\boldsymbol{A}$ has no distinguished constants.

Theorem 2.4. *$\langle A, \mathrm{Sub}(\boldsymbol{A})\rangle$ is an algebraic closed-set system for every Σ-algebra $\boldsymbol{A}$.*

Proof. Let $\mathcal{K} \subseteq \mathrm{Sub}(\boldsymbol{A})$. Let $\sigma \in \Sigma$ and $a_1, \dots, a_n \in \bigcap \mathcal{K}$, where $n = \rho(\sigma)$. Then for every $B \in \mathcal{K}$, $a_1, \dots, a_n \in B$ and hence $\sigma^{\boldsymbol{A}}(a_1, \dots, a_n) \in B$. So $\sigma^{\boldsymbol{A}}(a_1, \dots, a_n) \in \bigcap \mathcal{K}$. So $\bigcap \mathcal{K} \in \mathrm{Sub}(\boldsymbol{A})$.

Suppose now that $\mathcal{K}$ is directed. Let $a_1, \dots, a_n \in \bigcup \mathcal{K}$. Since there is only a finite number of the a_i, they are all contained in a single $B \in \mathcal{K}$. So $\sigma^{\boldsymbol{A}}(a_1, \dots, a_n) \in B \in \bigcup \mathcal{K}$. Hence $\bigcup \mathcal{K} \in \mathrm{Sub}(\boldsymbol{A})$. □

Note that if Σ is unary, then $\bigcup \mathcal{K} \in \mathrm{Sub}(\boldsymbol{A})$ for every $\mathcal{K} \subseteq \mathrm{Sub}(\boldsymbol{A})$ because $a \in \bigcup \mathcal{K}$ implies $a \in B$ for some $B \in A$, and hence $\sigma^{\boldsymbol{A}}(a) \in B \subseteq \bigcup \mathcal{K}$.

Note also that $\mathrm{Sub}(\boldsymbol{A}) = \boldsymbol{\mathcal{P}}(A)$, i.e., every subset of A is a subuniverse of $\boldsymbol{A}$, iff, for every $n \in \Sigma$, every $\sigma \in \Sigma$, and all $a_1, \dots, a_n \in A$, $\sigma^{\boldsymbol{A}}(a_1, \dots, a_n) \in \{a_1, \dots, a_n\}$.

The closure operator associated with the closed-set system $\langle A, \mathrm{Sub}(\boldsymbol{A})\rangle$ is denoted by $\mathrm{Sg}^{\boldsymbol{A}}$. Thus $\mathrm{Sg}^{\boldsymbol{A}} \colon \boldsymbol{\mathcal{P}}(A) \to \mathrm{Sub}(\boldsymbol{A})$ and $\mathrm{Sg}^{\boldsymbol{A}}(X) = \bigcap\{\, B \in \mathrm{Sub}(\boldsymbol{A}) : X \subseteq B \,\}$; this is called the *subuniverse generated* by X.

Theorem 2.5 (Birkhoff-Frink)**.** *Let $\langle A, \mathcal{C}\rangle$ be an algebraic closed-set system over A. There there exists a signature Σ and a Σ-algebra $\boldsymbol{A}$ such that $\mathcal{C} = \mathrm{Sub}(\boldsymbol{A})$.*

Proof. For every $X \subseteq_\omega A$ and every $b \in \mathrm{Cl}_{\mathcal{C}}(X)$, let $\sigma_{X,b}$ be an operation symbol of rank $|X|$, the cardinality of X. Let $\Sigma = \{\, \sigma_{X,b} : X \subseteq_\omega A, b \in \mathrm{Cl}_{\mathcal{C}}(X) \,\}$. Let $\boldsymbol{A}$ be the Σ-algebra with universe A such that, for every $\sigma_{X,b} \in \Sigma$ and all $a_1, \dots, a_n \in A$, where $n = |X|$,

$$\sigma^{\boldsymbol{A}}(a_1, \dots, a_n) = \begin{cases} b & \text{if } \{a_1, \dots, a_n\} = X \\ a_1 & \text{otherwise.} \end{cases}$$

If $X = \emptyset$, $\sigma^{\boldsymbol{A}}_{X,b} = b$.

In order to establish the conclusion of the theorem it suffices to show that,

$$\text{for every } Y \subseteq A,\ \mathrm{Cl}_{\mathcal{C}}(Y) = \mathrm{Sg}^{\boldsymbol{A}}(Y).$$

$\supseteq$: We first show that $\mathrm{Cl}_{\mathcal{C}}(Y)$ is closed under the operations of $\boldsymbol{A}$. Let $X \subseteq_\omega A$ and $b \in \mathrm{Cl}_{\mathcal{C}}(X)$. Let $a_1, \dots, a_n \in \mathrm{Cl}_{\mathcal{C}}(Y)$, were $n = |X|$. If $\{a_1, \dots, a_n\} = X$, then $\sigma^{\boldsymbol{A}}_{X,b}(a_1, \dots, a_n) = b \in \mathrm{Cl}_{\mathcal{C}}(X) \subseteq \mathrm{Cl}_{\mathcal{C}}(Y)$, since $X \subseteq \mathrm{Cl}_{\mathcal{C}}(Y)$; otherwise, $\sigma^{\boldsymbol{A}}_{X,b}(a_1, \dots, a_n) = a_1 \in \mathrm{Cl}_{\mathcal{C}}(Y)$. So $Y \subseteq \mathrm{Cl}_{\mathcal{C}}(Y) \in \mathrm{Sub}(\boldsymbol{A})$, and hence $\mathrm{Sg}^{\boldsymbol{A}}(Y) \subseteq \mathrm{Cl}_{\mathcal{C}}(Y)$.

$\subseteq$. Let $b \in \mathrm{Cl}_{\mathcal{C}}(Y)$. Since $\langle A, \mathcal{C}\rangle$ is a algebraic by hypothesis, $\mathrm{Cl}_{\mathcal{C}}(Y) = \bigcup\{\, \mathrm{Cl}_{\mathcal{C}}(Y') : Y' \subseteq_\omega Y \,\}$. So there is a finite subset Y' of Y such that $b \in \mathrm{Cl}_{\mathcal{C}}(Y')$. If $Y' = \emptyset$, then b

is in every subuniverse of $\boldsymbol{A}$, in particular in $\mathrm{Sg}^{\boldsymbol{A}}(Y)$. So assume $Y \neq \emptyset$, and let $Y' = \{a_1, \dots, a_n\}$, where the $a_1, \dots, a_n$ are all distinct. Then $b = \sigma_{Y',b}(a_1, \dots, a_n) \in \mathrm{Sg}^{\boldsymbol{A}}(Y)$. So $\mathrm{Cl}_{\mathcal{C}}(Y) \subseteq \mathrm{Sg}^{\boldsymbol{A}}(Y)$. □

We examine the subuniverse lattices of some familiar algebras. Let $\mathbb{Z} = \langle \mathbb{Z}, +, -, 0 \rangle$. We begin by showing that

$$\mathrm{Sub}(\mathbb{Z}) = \{\, n\mathbb{Z} : n \in \omega \,\}, \tag{19}$$

where $n\mathbb{Z} = \{n \cdot k : k \in \mathbb{Z}\,\}$ and $\cdot$ is integer multiplication.

Let $H \in \mathrm{Sub}(\mathbb{Z})$. If $H = \{0\}$, then $H = 0\mathbb{Z}$. Assume $H \neq \{0\}$. Then $H \cap \omega \neq \{0\}$, because if $k \in H$ and $h < 0$, then $-k \in H$. Let n be the least element of $H \cap (\omega \setminus \{0\})$. Let $k \in \mathbb{Z}$.

- If $k = 0$, $n \cdot k = 0 \in H$.
- If $k > 0$, $n \cdot k = \underbrace{n + n + \cdots + n}_{k} \in H$.
- If $k < 0$, $n \cdot k = \underbrace{-n + -n + \cdots + -n}_{-k} \in H$.

So $n\mathbb{Z} \subseteq H$.

Suppose $k \in H$. By the division algorithm, $k = qn + r$, where $0 \leq r < n$. $r = k - qn \in H \cap \omega$. So $r = 0$ by the minimality of n. Thus $k \in n\mathbb{Z}$, and $H \subseteq n\mathbb{Z}$. This verifies (19).

Let $\mathbf{Sub}(\mathbb{Z})$ be the complete lattice $\langle \mathrm{Sub}(\mathbb{Z}), \vee, \cap \rangle$. We note that $n\mathbb{Z} \vee m\mathbb{Z} = \{\, qn + pm : p, q \in \mathbb{Z}\,\} = \{\mathrm{GCD}(n, m)r : r \in \mathbb{Z}\,\}$. So

$$n\mathbb{Z} \vee n\mathbb{Z} = \mathrm{GCD}(n, m)\mathbb{Z}.$$

It is left as an exercise to show that $n\mathbb{Z} \wedge m\mathbb{Z} = \mathrm{LCM}(n, m)\mathbb{Z}$. Thus $\mathbf{Sub}(\mathbb{Z}) \cong \langle \omega, \mathrm{GCD}, \mathrm{LCM} \rangle$.

Note that $\mathrm{Sub}(\langle \omega, S \rangle = \{\, [n) : n \in \omega\,\} \cup \{\emptyset\}$, where S is the successor function, i.e., $S(n) = n + 1$, and $[n) = \{\, k \in \omega : n \leq k\,\}$. We have $[n) \vee [m) = [n) \cup [m) = [\mathrm{Min}(n, m))$ and $[n) \cap [m) = [\mathrm{Max}(n, m))$. Thus $\big\langle \mathrm{Sub}(\langle \omega, S \rangle), \vee, \cap \big\rangle \cong \langle \omega \cup \{\infty\}, \mathrm{Min}, \mathrm{Max} \rangle$. So $\big\langle \mathrm{Sub}(\langle \omega, S \rangle), \subseteq \big\rangle \cong \langle \omega \cup \{\infty\}, \geq \rangle$

$\mathrm{Sub}(\langle \omega, P \rangle = \{\, [0, n] : n \in \omega\,\} \cup \{\emptyset\}$, where P is the predecessor function, i.e., $P(n) = n - 1$ if $0 < n$; $P(0) = 0$. $\big\langle \mathrm{Sub}(\langle \omega, P \rangle), \vee, \cap \big\rangle \cong \langle \omega \cup \{\infty\}, \mathrm{Max}, \mathrm{Min} \rangle$. So $\big\langle \mathrm{Sub}(\langle \omega, P \rangle), \subseteq \big\rangle \cong \langle \omega \cup \infty, \leq \rangle$

If $\boldsymbol{A} = \langle A, \vee, \wedge \rangle$ is a lattice, $\langle A, \wedge, \vee \rangle$ is also a lattice, called the *dual* of $\boldsymbol{A}$. Its Hasse diagram is obtained by turning the Hasse diagram of $\boldsymbol{A}$ up-side-down. The lattices of subuniverses of $\langle \omega, S \rangle$ and $\langle \omega, P \rangle$ are duals of each other.

A lattice is *bounded* if it has a largest element, i.e., an element that is an upper bound of every element of the lattice, and also a smallest element. These elements are normally denoted by 1 and 0, respectively. The elements of the lattice that cover 0 (if any exist) are called *atoms*, and the elements that are covered by 1 are called *coatoms*. The coatoms of the lattice $\mathbf{Sub}(\boldsymbol{A})$ of subuniverses of a Σ-algebra $\boldsymbol{A}$ are called *maximal proper* subuniverses. Thus B is a maximal proper subuniverse of $\boldsymbol{A}$ if $B \neq A$ and there does not exist a $C \in \mathrm{Sub}(\boldsymbol{A})$ such that $B \subsetneq C \subsetneq A$.

The maximal proper subuniverses $\mathbb{Z}$ are of the form $p\mathbb{Z}$, p a prime. The only maximal proper subuniverse of $\langle \omega, S\rangle$ is $[1)$, while $\langle \omega, P\rangle$ has no maximal proper subuniverse. This is a reflection of the fact that $\mathbb{Z}$ and $\langle \omega, S\rangle$ are both finitely generated, in fact by 1 and 0, respectively, while $\langle \omega, P\rangle$ is not finitely generated. This connection between the existence of maximal proper subuniverses and finite generation is a general phenomenon as the following theorem shows. Note that since every natural number greater than 1 has a prime factor, every proper subuniverse of $\mathbb{Z}$ is included in a maximal proper subuniverse, and trivially $\langle \omega, S\rangle$ has the same property.

Theorem 2.6. *Let $\boldsymbol{A}$ be a finitely generated Σ-algebra. Then every proper subuniverse of $\boldsymbol{A}$ is included in a maximal proper one.*

Let B be a proper subuniverse of $\boldsymbol{A}$. The theorem is obvious if $\boldsymbol{A}$ is finite: If B is not maximal, let B' be a proper subuniverse that is strictly larger than B. If B' is not maximal, let B'' be a proper subuniverse that is strictly larger than B'. Continue in this way. If $|A| = n$, this process cannot continue for more than n steps. If B is infinite, the process may continue ω steps (here we are thinking of ω as a ordinal number). In order to prove the theorem in general we need to be able to extend the process beyond ω steps to the transfinite. *Zorn's lemma*[1] allows us to do this. Let $\langle A, \leq\rangle$ be a (nonempty) poset with the property that every chain (i.e., linearly ordered subset) has an upper bound in A. Then Zorn's lemma asserts that $\langle P, \leq\rangle$ has a maximal element.

We are now ready to prove the theorem.

Proof. Let $A = \mathrm{Sg}^{\boldsymbol{A}}(X)$, $X \subseteq_\omega A$. Let $B \in \mathrm{Sub}(\boldsymbol{A})$, $B \neq A$. Let $\mathcal{K} = \{\, K \in \mathrm{Sub}(\boldsymbol{A}) : B \subseteq K \subsetneq A \,\}$. $\mathcal{K}$ is nonempty since it contains B. Let $\mathcal{C} \subseteq \mathcal{K}$ be any chain. $\mathcal{C}$ is directed, so $\bigcup \mathcal{C} \in \mathrm{Sub}(\boldsymbol{A})$. $\bigcup \mathcal{C}$ is a proper subuniverse, because if $\bigcup \mathcal{C} = A$, then $X \subseteq \bigcup \mathcal{C}$, and hence $X \subseteq K$ for some $K \in \mathcal{K}$, because X is finite and $\mathcal{C}$ is directed. But this is impossible since $X \subseteq K$ implies $K = A$ and hence $K \notin \mathcal{K}$. So every chain in $\mathcal{K}$ has an upper bound. By Zorn's lemma $\mathcal{K}$ has a maximal element. $\square$

The converse of the theorem does not hold: there are algebras that are not finitely generated but which still have maximal proper subuniverses. For example, let $\boldsymbol{A} = \langle \omega \cup \{\infty\}, P\rangle$, where P is the usual predecessor function on ω and $P(\infty) = \infty$. $\boldsymbol{A}$ is clearly not finitely generated, but ω is a maximal proper subuniverse.

Theorem 2.7 (Principle of Structural Induction)**.** *Let $\boldsymbol{A}$ be a Σ-algebra generated by X. To prove that a property $\mathcal{P}$ holds for each element of $\boldsymbol{A}$ it suffices to show that*

(i) induction basis. *$\mathcal{P}$ holds for each element of X.*
(ii) induction step. *If $\mathcal{P}$ holds for each of the elements $a_1, \dots, a_n \in A$ (the* induction hypothesis*), then $\mathcal{P}$ holds for $\sigma^{\boldsymbol{A}}(a_1, \dots, a_n)$ for all $\sigma \in \Sigma_n$.*

Proof. Let $P = \{\, a \in A : \mathcal{P} \text{ holds for } a \,\}$. $X \subseteq P$ and P is closed under the operations of $\boldsymbol{A}$. So $P \in \mathrm{Sub}(\boldsymbol{A})$. Hence $A = \mathrm{Sg}^{\boldsymbol{A}}(X) \subseteq P$. $\square$

Ordinary mathematical induction is the special case $\boldsymbol{A} = \langle \omega, S\rangle$. $\omega = \mathrm{Sg}^{\boldsymbol{A}}(\{0\})$. If 0 has the property $\mathcal{P}$ and n has $\mathcal{P}$ implies $S(n) = n + 1$ has $\mathcal{P}$, the every natural number has the property $\mathcal{P}$.

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[1] Zorn's lemma is a theorem of set theory that is know to be equivalent to the Axiom of Choice, and hence independent of the usual axioms of set theory. It is of course highly nonconstructive.

We now show how the notion of subuniverse extends naturally for single-sorted to multi-sorted signatures. Let Σ be a multi-sorted signature with set S of sorts. Let $\boldsymbol{A} = \big\langle \langle\, A_s : s \in S \,\rangle, \sigma^{\boldsymbol{A}} \big\rangle_{\sigma \in \Sigma}$ be a Σ-algebra. A *subuniverse* of $\boldsymbol{A}$ is an S-sorted set $B = \langle\, B_s : s \in S \,\rangle$ such that $B_s \subseteq A_s$ for every $s \in S$, and, for every $\sigma \in \Sigma$ of type $s_1, \dots, s_n \to t$, and all $a_1 \in A_{s_1}, \dots, a_n \in A_{s_n}$, we have $\sigma^{\boldsymbol{A}}(a_1, \dots, a_n) \in B_t$.

$\langle \mathrm{Sub}(\boldsymbol{A}), \leq \rangle$ is a complete lattice where $B = \langle B_s : s \in S \rangle \leq C = \langle C_s : s \in S \rangle$ iff $B_s \subseteq C_s$ for all $s \in S$. The (infinite) join and meet operations are:

$$\bigvee \mathcal{K} = \big\langle \bigcap \{\, B_s : B \in \mathcal{K} \,\} : s \in S \,\big\rangle,$$

$$\bigwedge \mathcal{K} = \big\langle \bigcap \{\, C_s : \ C \in \mathrm{Sub}(\boldsymbol{A}) \text{ such that, for all } B \in \mathcal{K},\ B \leq C \,\} : s \in S \,\big\rangle.$$

As an example consider the algebra

$$\mathbf{Lists}(\omega) = \big\langle \langle \omega \cup \{e_D\}, \omega^* \cup \{e_l\} \rangle, \mathtt{head}, \mathtt{tail}, \mathtt{append}, \mathtt{emptylist}, \mathtt{derror}, \mathtt{derror} \big\rangle.$$

We leave it as an exercise to show that every subuniverse is generated by a unique subset of the data set ω; more precisely, the subuniverses of **Lists** are exactly the S-sorted subsets of Lists of the form $\langle X_D \cup \{e_D\},\ X_D^* \cup \{e_L\} \rangle$ were X is an arbitrary subset of ω.

Hint: It is easy to check that this is a subuniverse. For the other direction it suffices to show that for any sorted set $X = \langle X_D, X_L \rangle \leq \langle \omega \cup \{e_D\}, \omega^* \cup \{e_l\} \rangle$, $\mathrm{Sg}^{\mathbf{Lists}(\omega)}(X) = \mathrm{Sg}^{\mathbf{Lists}(\omega)}(\langle Y, \emptyset \rangle)$, where $y = X \cup \bigcup \{\, \{a_1, \dots, a_n\} : \langle a_1, \dots, a_n \rangle \in X_L \,\}$. For this purpose it suffices to show that, if $\langle a_1, \dots, a_n \rangle \in X_L$, then $\{a_1, \dots, a_n\} \subseteq B_D$, where $B = \mathrm{Sg}^{\mathbf{Lists}(\omega)}(X)$. But, for all $i \leq n$, $a_i = \mathtt{head}(\mathtt{tail}^i(\langle a_1, \dots, a_n \rangle)) \in B_D$.

From this characterization of the subuniverses of **Lists** is follows easily that **Sub**(**Lists**) is isomorphic to the lattice of all subsets of ω and hence is distributive.