Definition 1.12. A poset $\boldsymbol{A} = \langle A, \leq \rangle$ is *complete* if, for every $X \subseteq A$, $\mathrm{LUB}(X)$ and $\mathrm{GLB}(X)$ both exist.

We denote $\mathrm{LUB}(X)$ by $\bigvee X$ (if it exists) and $\mathrm{GLB}(X)$ by $\bigwedge X$.

If $\boldsymbol{A}$ is a lattice and $X = \{x_1, \dots, x_n\}$ is finite, then $\mathrm{LUB}(X)$ and $\mathrm{GLB}(X)$ always exist and equal $x_1 \vee \cdots \vee x_n$ and $x_1 \wedge \cdots \wedge x_n$, respectively. Thus every finite lattice is complete.

$\langle \omega, \leq \rangle$, $\langle \mathbb{Q}, \leq \rangle$ ($\mathbb{Q}$ the rational numbers), and $\langle \mathbb{R}, \leq \rangle$ ($\mathbb{R}$ the real numbers) are not complete, but the natural numbers and the real numbers can be made complete by adjoining a largest element ∞ to the natural numbers and both a smallest element $-\infty$ and a largest element $+\infty$ to the reals. The rationals cannot be completed so easily; in fact, for every irrational number r, $\{\, q \in \mathbb{Q} : q < r \,\}$ fails to have a least upper bound. For any set A, $\langle \boldsymbol{\mathcal{P}}(A), \subseteq \rangle$ is a complete lattice. $\bigvee K = \bigcap K$ and $\bigwedge K = \bigcup K$ for every $K \subseteq \boldsymbol{\mathcal{P}}(A)$.

Theorem 1.13. *Let $\boldsymbol{A} = \langle A, \leq \rangle$ be a poset. For every $X \subseteq A$, $\bigvee X$ exists iff, for every $X \subseteq A$, $\bigwedge X$ exists. Thus a poset and in particular a lattice is complete iff every subset has a* LUB*, equivalently, iff every subset has a* GLB.

Proof. $\Longrightarrow$. Assume $\mathrm{LUB}(X)$ exists for every $X \subseteq A$. It suffices to show that, for every $X \subseteq A$, $\mathrm{GLB}(X) = \mathrm{LUB}(\mathrm{LB}(X))$; recall that $\mathrm{LB}(X) = \{\, y \in A : \text{for every } x \in X,\ y \leq x \,\}$, the set of lower bounds of X. Let $a = \mathrm{LUB}(\mathrm{LB}(X))$. For every $x \in X$, $x \in \mathrm{UB}(\mathrm{LB}(X))$. Thus, for every $x \in X$, $x \geq \mathrm{LUB}(\mathrm{LB}(X)) = a$. So $a \in \mathrm{LB}(X)$, and clearly, for every $y \in \mathrm{LB}(X)$, $a \geq y$.

$\Longleftarrow$. The proof is the dual of the one above, i.e., it is obtained by interchanging "$\leq$" and "$\geq$", "LUB" and "GLB", and "UB" and "LB". □

For any group $\boldsymbol{G}$, $\mathbf{Sub}(\boldsymbol{G}) = \langle \mathrm{Sub}(\boldsymbol{G}), \subseteq \rangle$ is a complete lattice. For every $K \subseteq \mathrm{Sub}(\boldsymbol{G})$, $\bigvee K = \bigcap K$ and $\bigwedge K = \bigcap \{\, H \in \mathrm{Sub}(\boldsymbol{G}) : \bigcup K \subseteq H \,\}$.

Why doesn't $\langle \omega, \leq \rangle$ contradict the above theorem? Isn't the greatest lower bound of any set of natural numbers the smallest natural number in the set? This is true for any *nonempty* set, but $\emptyset$ has no GLB in ω. For any poset $\boldsymbol{A}$, $\mathrm{GLB}(\emptyset)$ (if it exists) is the largest element of $\boldsymbol{A}$, and $\mathrm{LUB}(\emptyset)$ is the smallest element of $\boldsymbol{A}$.

Let $\boldsymbol{A} = \langle \omega \cup \{a, b, \infty\}, \leq \rangle$, where $\leq$ is the natural order on ω, for every $n \in \omega$, $n < a, b, \infty$, and $a, b < \infty$. Every finite subset of A has a LUB (including $\emptyset$), but $\mathrm{GLB}(a, b)$ does not exist. So the requirement that X ranges over *all* subsets of A in the theorem is critical. But if $\boldsymbol{A}$ is a finite poset, and $\mathrm{LUB}(a, b)$ exists for every pair of elements of $\boldsymbol{A}$ and $\boldsymbol{A}$ has a smallest element, then $\boldsymbol{A}$ is a complete lattice.

Some notation: if $\boldsymbol{A}$ is a complete lattice, $1 = \bigvee A = \bigwedge \emptyset$ will denote the largest element of $\boldsymbol{A}$, and $0 = \bigwedge A = \bigvee \emptyset$ will denote the smallest element.

Definition 1.14. Let $\boldsymbol{A} = \langle A, \vee, \wedge \rangle$ and $\boldsymbol{B} = \langle B, \vee, \wedge \rangle$ be complete lattices. $\boldsymbol{B}$ is a *complete sublattice* of $\boldsymbol{A}$ if, for every $X \subseteq A$, $\bigvee^{\boldsymbol{B}} X = \bigvee^{\boldsymbol{A}} X$ and $\bigwedge^{\boldsymbol{B}} X = \bigwedge^{\boldsymbol{A}} X$.

$\langle \{-2\} \cup (-1, +1) \cup \{+2\}, \leq \rangle$ is a complete lattice and a sublattice of the complete lattice $\langle \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}, \leq \rangle$ but not a complete sublattice.

Definition 1.15. Let A be a set. $E \subseteq A^2$ is an *equivalence relation* on A if

(E1) $$x\,R\,x,$$

(E2) $$x\,E\,y \text{ and } y\,E\,z \text{ imply } x\,E\,z,$$

(E3) $$x\,E\,y \text{ implies } y\,E\,x. \quad (\textit{symmetric law})$$

$\mathrm{Eq}(A)$ will denote the set of all equivalence relations on A. Let $K \subseteq \mathrm{Eq}(A)$.

(6) $$\bigcap K \in \mathrm{Eq}(A).$$

Check (E1)–(E3).

(E2). Assume $\langle a,b\rangle, \langle b,c\rangle \in \bigcap K$. For every $E \in K$, $\langle a,b\rangle, \langle b,c\rangle \in E$. Thus, for every $E \in K$, $\langle a,c\rangle \in E$. So $\langle a,c\rangle \in \bigcap K$. (E1) and (E3) are verified similarly.

So $\langle \mathrm{Eq}(A), \subseteq\rangle$ is a complete lattice with $\bigvee K = \bigcap K$ and

(7) $$\bigwedge K = \bigcap\{\, E \in \mathrm{Eq}(A) : \bigcup K \subset E \,\}.$$

The smallest equivalence relation on A is the *identity* or *diagonal relation*, $\Delta_A = \{\, \langle a,a\rangle : a \in A \,\}$, read "delta A". The largest equivalence relation is the *universal relation*, $\nabla_A = A \times A$, read "nabla A".

The description of the join operation in (7) is what can be called a "coinductive" or "from above" characterization; it is very useful for theoretical purposes, for example proving general propositions about the join, but it does not give much information about what the elements of the join of K look like in terms of the elements of the subgroups or the ordered pairs of the equivalence relations of K. For this we need an "inductive" or "from below" characterization.

Theorem 1.16. *Let H, K be subgroups of $\boldsymbol{G} = \langle G, \cdot, ^{-1}, e\rangle$.*

$$H \vee K = HK \cup HKHK \cup HKHKHK \cup \cdots = \bigcup_{1 \le n \in \omega} (HK)^n,$$

where $(HK)^n = \{\, h_1 \cdot k_1 \cdots h_n \cdot k_n : h_1, \dots, h_n \in H, k_1, \dots, k_n \in K \,\}$.

Proof. Let $L = \bigcup_{1 \le n \in \omega} (HK)^n$. We must show three things.

(8) $$L \in \mathrm{Sub}(\boldsymbol{G}),$$

(9) $$H, K \subseteq L,$$

(10) $$\text{for all } M \in \mathrm{Sub}(\boldsymbol{G}),\ H, K \subseteq M \text{ implies } L \subseteq M.$$

Proof of (8). Clearly $L \neq \emptyset$. Let $a, b \in L$. We must show $ab \in L$ (following convention we often omit the "$\cdot$" when writing the product of elements of groups) and $a^{-1} \in L$. $a \in (HK)^n$ and $b \in (HK)^m$ for some $n, m \in \omega$. So $ab \in (HK)^{n+m} \subseteq L$, and $a^{-1} \in K^{-1}H^{-1} \cdots K^{-1}H^{-1} = (KH)^n = \{e\}(KH)^n\{e\} = H(KH)^nK = (HK)^{n+1} \subseteq L$.

(9). $H = H\{e\} \subseteq HK \subseteq L$ and $K \subseteq \{e\}K \subseteq HK \subseteq L$.

(10). Suppose $H, K \subseteq L \in \mathrm{Sub}(\boldsymbol{G})$. We prove by induction on n that $(HK)^n \subseteq M$. $HK \subseteq MM \subseteq M$. Assume $(HK)^n \subseteq M$. Then $(HK)^{n+1} \subseteq (HK)^nHK \subseteq MM \subseteq M$. $\square$

A similar argument gives a inductive characterization of the join of an arbitrary set of subgroups $\{ H_i : i \in I \}$. We leave as an exercise the proof that

$$\bigvee_{i\in I} H_i = \bigcup_{\langle h_1,\dots,h_n\rangle \in I^*} H_{i_1} \cdots H_{i_n},$$

where I^* is the set of all finite sequences of elements of I.

We next obtain a similar "inductive" characterization of the join of equivalence relations. For this purpose we need to explain some basic results in the "calculus of relations". Let A, B, C be arbitrary sets and $R \subseteq A \times B$ and $S \subseteq B \times C$. By the *relative product* of R and S, in symbols $R \circ S$, we mean the relation

$$\{ \langle a, c\rangle \in A \times C : \text{there exists a } b \in B \text{ such that } a\, R\, b\, S\, c \}.$$

The relative product is a binary operation on the set of all binary relations on any set A such that Δ_A acts as an identity (i.e., $\Delta_A \circ R = R \circ \Delta_A = R$). Also ∇_A acts like an infinity element on reflexive relations, i.e., $\Delta_A \subseteq A$ implies $\nabla_A \circ R = R \circ \nabla_A = \nabla_A$. We also have a unary *converse* operation that has some of the properties of the inverse of a group (but is not a group inverse). $\check{R} = \{ \langle a, b\rangle : \langle b, a\rangle \in R \}$, i.e., $a\, \check{R}\, b$ iff $b\, \check{R}\, a$. $a\, (R \circ S)^{\smile}\, a'$ iff $a'\, (R \circ S)\, a$ iff there exists a b such that $a'\, R\, b\, S\, a$ iff there exists a b such that $a\, \check{S}\, b\, \check{R}\, a'$ iff $a\, \check{S} \circ \check{R}\, a'$. So $(R \circ S)^{\smile} = \check{S} \circ \check{R}$.

We note that the notation of the calculus of relations can be used to formulate the defining conditions of an equivalence relation in very simple terms. The reflexive law: $\Delta_A \subseteq E$; the transitive law: $E \circ E \subseteq E$; the symmetric law: $\check{E} \subseteq E$.

Theorem 1.17. *Let $E, F \in \mathrm{Eq}(A)$.*

$$E \vee F = E \circ F \cup E \circ F \circ E \circ F \cup \cdots = \bigcup_{1 \le n \in \omega} (E \circ F)^n.$$

Proof. Let $G = \bigcup_{1\le n\in\omega} (E \circ F)^n$. We show that $G \in \mathrm{Eq}(A)$. $\Delta_A = \Delta_A \circ \Delta_A \subseteq E \circ F \subseteq G$. Assume $\langle a, b\rangle, \langle b, c\rangle \in G$, i.e., that there exist $n, m \in \omega$ such that $a\, (E \circ F)^n\, b\, (E \circ F)^m\, c$. Then $\langle a, c\rangle \in (E \circ F)^n \circ (E \circ F)^m = (E \circ F)^{n+m} \subseteq G$. We also have that $\langle b, a\rangle \in \big((E \circ F)^n\big)^{\smile} = (\check{F} \circ \check{E})^n = (F \circ E)^n \subseteq (E \circ F)^{n+1} \subseteq G$. The proof that $G = E \vee F$ is left at an exercise. □

We also leave as an exercise the following inductive characterization of the join of an arbitrary set $\{ E_i : i \in I \}$ of equivalence relations on a set A.

$$\bigvee_{i\in I} E_i = \bigcup_{\langle h_1,\dots,h_n\rangle \in I^*} E_{i_1} \circ \cdots \circ E_{i_n}.$$

Exercise. Let $R \subseteq A^2$ be an arbitrary binary relation on A. Prove that $\bigcup_{1\le n\in\omega} R^n$, where $R^n = R \circ R \circ \cdots \circ R$ with n repetitions of R, is the smallest transitive relation that includes R. It is called the *transitive closure* of R.

Every equivalence relation is uniquely determined by its corresponding partition.

$\mathcal{P}$ is a *partition* of a set A if

- $\mathcal{P} \subseteq \boldsymbol{\mathcal{P}}(A) \setminus \{\emptyset\}$,
- $\bigcup \mathcal{P}$ $(= \bigcup\{ X : X \in \mathcal{P} \}) = A$,
- for all $X, Y \in \mathcal{P}$, $X \ne Y$ implies $X \cap Y = \emptyset$.

Let $E \in \mathrm{Eq}(A)$. For each $a \in A$, let $[a]_E = \{\, x \in A : x \mathrel{E} a \,\}$, called the *equivalence class of* a (*over* E); $[a]_E$ is also denoted by a/E.

$\{\, [a]_E : a \in A \,\}$ is a partition of A (exercise). Conversely, if $\mathcal{P}$ is a partition of A, define $a \equiv_{\mathcal{P}}$ by the condition that there is an $X \in \mathcal{P}$ such that $a, b \in X$. Then $\equiv_{\mathcal{P}}$ is an equivalence relation whose partition is $\mathcal{P}$ (exercise). Moreover, for each equivalence relation E, $\equiv_{\{\, [a]_E : a \in A \,\}} = E$.

$\mathrm{Part}(A)$ denotes the set of partitions of A, and $\mathbf{Part}(A) = \langle \mathrm{Part}(A), \leq \rangle$, where $\leq$ is the partial ordering on $\mathrm{Part}(A)$ defined as follows: $\mathcal{P} \leq \mathcal{Q}$ if, for each $X \in \mathcal{P}$, there exists a $Y \in \mathcal{Q}$ such that $X \subseteq Y$, equivalently, each equivalence class of $\mathcal{Q}$ is a union of equivalence classes of $\mathcal{P}$. The mapping $\mathcal{P} \mapsto \equiv_{\mathcal{P}}$ is bijection between the posets $\mathbf{Eq}(A)$ and $\mathbf{Part}(A)$ that is strictly order-preserving (exercise). Thus $\mathbf{Part}(A)$ is a complete lattice and the above mapping is a lattice isomorphism.

It is usually easier to picture the partition of a specific equivalence relation rather than the relation itself. The following characterizations of the join and meet operations in $\mathbf{Part}(A)$ are left as exercises. In the lattice $\mathbf{Part}(A)$,

$$\mathcal{P} \wedge \mathcal{Q} = \{\, X \cap Y : X \in \mathcal{P}, Y \in \mathcal{Q}, X \cap Y \neq \emptyset \,\}.$$

A finite sequence $X_1, Y_1, \dots, X_n, Y_n$, where $X_1, \dots, X_n \in \mathcal{P}$ and $Y_1, \dots, Y_n \in \mathcal{P}$ is called *connected* if $X_i \cap Y_i \neq \emptyset$ for all $i \leq n$ and $Y_i \cap X_{i+1} \neq \emptyset$ for all $i < n$. Exercise: show that, for every $a \in A$, $b \in [a]_{\mathcal{P} \cup \mathcal{Q}}$ iff there exists a connected sequence $X_1, Y_1, \dots, X_n, Y_n$ such that $a \in X_1$ and $b \in Y_n$.

Exercise. For each $n \in \omega \setminus \{0\}$, define $\equiv \pmod{n} \in \mathbb{Z}^2$ by $a \equiv b \pmod{n}$ if $a = b + kn$ for some $k \in \mathbb{Z}$, i.e., $n|a - b$. Show $\equiv \pmod{n} \in \mathrm{Eq}(\mathbb{Z})$. Describe the partition of $\equiv \pmod{n}$. Describe $\equiv \pmod{n} \wedge \equiv \pmod{m}$ and $\equiv \pmod{n} \vee \equiv \pmod{m}$.

The lattices of subgroups and equivalence relations have special properties. In the sequel we write $X' \subseteq_\omega X$ to mean that X' is a *finite* subset of X.

Definition 1.18. Let $\boldsymbol{A}$ be a lattice. An element c of $\boldsymbol{A}$ is *compact* if, for every $X \subseteq A$ such that $\bigvee X$ exists,

$$c \leq \bigvee X \text{ implies there exists a } X' \subseteq_\omega X \text{ such that } c \leq \bigvee X'.$$

The set of compact elements of $\boldsymbol{A}$ is denoted by $\mathrm{Comp}(\boldsymbol{A})$.

$\boldsymbol{A}$ is *compactly generated* if every element of $\boldsymbol{A}$ is the join of the compact elements less than or equal to it, i.e., for every $a \in A$,

$$a = \bigvee \{\, c \in \mathrm{Comp}\, \boldsymbol{A} : c \leq a \,\}.$$

A lattice is *algebraic* if it is complete and compactly generated.

We note that an element is compactly generated iff it is the join of some set of compact elements, since in this case it must be the join of all compact elements less than or equal to it.

Examples.

- Every finite lattice is algebraic.
- The lattice $\langle \omega \cup \{\infty\}, \leq \rangle$ is algebraic. ∞ is the only noncompact element and $\infty = \bigvee \omega$.

- $\langle [0,1], \leq \rangle$, where $[0,1] = \{ x \in \mathbb{R} : 0 \leq x \leq 1 \}$, is complete but not algebraic; 0 is the only compact element and hence the only element that is the join of compact elements.
- $\langle \mathrm{Sub}(\boldsymbol{G}), \subseteq \rangle$, for every group $\boldsymbol{G}$, and $\langle \mathrm{Eq}(A), \subseteq \rangle$, for every nonempty set A, are algebraic lattices, but this will be shown until later.

1.2. **Closed set systems and closure operators.** A family K of sets is said to be *upward directed by inclusion*, or *upward directed* for short, or even shorter, simply *directed*, if each pair of sets in K is included in a third member of K, i.e.,

$$\text{for all } X, Y \in K \text{ there exists a } Z \in K \text{ such that } X \vee Y \subseteq Z.$$

Definition 1.19. A *closed set system* consists of a nonempty set A and a $\mathcal{C} \subseteq \boldsymbol{\mathcal{P}}(A)$ such that $\mathcal{C}$ is closed under intersections of arbitrary subsets, i.e.,

$$\text{for every } K \subseteq \mathcal{C}, \bigcap K \in \mathcal{C}.$$

A closed set system $\langle A, \mathcal{C} \rangle$ is *algebraic* if $\mathcal{C}$ is closed under unions of upward directed subsets, i.e.,

$$\text{for every directed } K \subseteq \mathcal{C}, \bigcup K \in \mathcal{C}.$$

Note that by definition $\mathcal{C}$ always contains A since $A = \bigcap \emptyset$. Since $\mathrm{Sub}(\boldsymbol{G})$ and $\mathrm{Eq}(A)$ are closed under intersections of arbitrary subsets, to show they form algebraic closed-set systems it suffices to show they are closed under unions of directed subsets.

The union of any, not necessarily directed, $K \subseteq \mathrm{Sub}(\boldsymbol{G})$ contains the identity and is closed under inverse. Assume that K is directed. Let $a, b \in \bigcup K$, and let $H, L \in K$ such that $a \in H$ and $b \in L$. Choose $M \in K$ such that $K \cup L \subseteq M$. Then $ab \in M \subseteq \bigcup K$. So $\bigcup K \in \mathrm{Sub}(\boldsymbol{G})$.

The union of any, not necessarily directed, $K \subseteq \mathrm{Eq}(A)$ includes Δ_A and is closed under converse. Assume that K is directed. Let $\langle a, b \rangle, \langle b, c \rangle \in \bigcup K$, and let $H, L \in K$ such that $\langle a, b \rangle \in H$ and $\langle b, c \rangle \in L$. Choose $M \in K$ such that $K \cup L \subseteq M$. Then $\langle a, c \rangle \in M \subseteq \bigcup K$. So $\bigcup K \in \mathrm{Eq}(A)$.

Each of the defining conditions of a group involves only a finite number of group elements, and, similarly, each of the conditions that define equivalence relations involves only a finite number of ordered pairs. This is the common property that guarantees subgroups and equivalence relations form an algebraic lattice. This vague observation will be made more precise in the next chapter.