

3.1. Identities, models, and varieties. We now change the focus of our investigations from individual algebras to classes of algebras, and in particular to the way these classes are defined. The axiomatic approach is a fundamental feature of modern algebra. For instance, while the number theorist focuses all his or her attention on the ring of integers, to the algebraist the integers are but one member, albeit the paradigm, of the class of principal ideal domains. More precisely, the algebraist is concerned less with the particular structure of the integers than with the abstract properties of the integers that give them their special nature. These abstract properties are formulated in a formal language, and the language itself, including the deductive apparatus by which the logical relationships between the propositions of the language are specified, then becomes the object of mathematical investigation. The mathematics of the language of mathematics is called *metamathematics*. In algebra many of the most important classes of algebras are defined by laws, or identities. Consequently we concentrate first on mathematics of the language of equations and the deductive apparatus associated with them. This is called *equational logic*.

From now on, unless specifically indicated otherwise, the symbol X will stand for $X_\omega = \{x_0, x_1, x_2, \dots\}$, a fixed countably infinite set of pairwise distinct variable symbols. Note that the numbering starts at zero. From now on all enumerated systems will be ordered by cardinals and hence the numbering will start with 0. In particular a finite sequence of length n will be written as a_0, \dots, a_{n-1} rather than as a_1, \dots, a_n .

Note that $\text{Te}_\Sigma(X) = \bigcup_{n \in \omega} \text{Te}_\Sigma(x_0, \dots, x_{n-1})$.

An Σ -equation, or simply an *equation* when the specific signature is not relevant, is an ordered pair $\langle t, s \rangle$ with $t, s \in \text{Te}_\Sigma(X)$. We will almost always write $\langle t, s \rangle$ in the form $t \approx s$ in order to exploit the intuitive meaning of equations. X represents a countably infinite “reservoir” of variable symbols to draw from, but in a particular term or equation, or more generally in any particular finite set of terms or equations, only a finite number of variables will actually occur. When we want to indicate explicitly the variables that may actually occur in an equation $t \approx s$ we often write $t(x_0, \dots, x_{n-1}) \approx s(x_0, \dots, x_{n-1})$. We emphasize that these two expressions represent exactly the same equation; the “ (x_0, \dots, x_{n-1}) ” should be look on as simply annotation in the metalanguage that imparts some additional information about the equation.

Definition 3.18. Let \mathbf{A} be a Σ -algebra.

- (i) A Σ -equation $t(x_0, \dots, x_{n-1}) \approx s(x_0, \dots, x_{n-1})$ is an *identity* of \mathbf{A} , and \mathbf{A} is a *model* of $t \approx s$, in symbols $\mathbf{A} \models t \approx s$, if, for all $a_0, \dots, a_{n-1} \in \mathbf{A}$, $t^{\mathbf{A}}(a_0, \dots, a_{n-1}) = s^{\mathbf{A}}(a_0, \dots, a_{n-1})$, i.e., for every $h^* \in \text{Hom}(\text{Te}_\Sigma(X), \mathbf{A})$, $h^*(t) = h^*(s)$.
- (ii) \mathbf{A} is a *model* of a set E of equations if, for every $t \approx s \in E$, $\mathbf{A} \models t \approx s$. The class of all models of E is denoted by $\text{Mod}(E)$.
- (iii) $t \approx s$ is an *identity* of a class \mathbf{K} of Σ -algebras if, for every $\mathbf{A} \in \mathbf{K}$, $\mathbf{A} \models t \approx s$. The set of all identities of \mathbf{K} is denoted by $\text{Id}(\mathbf{K})$.

Lemma 3.19. Let \mathbf{A} be a Σ -algebra and $t(x_0, \dots, x_{n-1}) \approx s(x_0, \dots, x_{n-1})$ a Σ -equation.

- (i) If $\mathbf{A} \models t \approx s$, then, for every $\mathbf{B} \subseteq \mathbf{A}$, $\mathbf{B} \models t \approx s$.
- (ii) If $\mathbf{A} \models t \approx s$, then, for every $\mathbf{B} \preceq \mathbf{A}$, $\mathbf{B} \models t \approx s$.

Let $\langle \mathbf{A}_i : i \in I \rangle$ be a system of Σ -algebras.

- (iii) If, for all $i \in I$, $\mathbf{A}_i \models t \approx s$, then $\prod_{i \in I} \mathbf{A}_i \models t \approx s$.

Proof. (i) For all $b_0, \dots, b_{n-1} \in B$, $t^{\mathbf{B}}(b_0, \dots, b_{n-1}) = t^{\mathbf{A}}(b_0, \dots, b_{n-1}) = s^{\mathbf{A}}(b_0, \dots, b_{n-1}) = s^{\mathbf{B}}(b_0, \dots, b_{n-1})$.

(ii) Let $h: \mathbf{A} \rightarrow \mathbf{B}$. Let $b_0, \dots, b_{n-1} \in B$ and choose $a_0, \dots, a_{n-1} \in A$ such that $h(a_i) = b_i$ for all $i \in I$. Then

$$\begin{aligned} t^{\mathbf{B}}(b_0, \dots, b_{n-1}) &= t^{\mathbf{B}}(h(a_0), \dots, h(a_{n-1})) \\ &= h(t^{\mathbf{A}}(a_0, \dots, a_{n-1})) \\ &= h(s^{\mathbf{A}}(a_0, \dots, a_{n-1})) \\ &= s^{\mathbf{B}}(h(a_0), \dots, h(a_{n-1})) \\ &= s^{\mathbf{B}}(b_0, \dots, b_{n-1}). \end{aligned}$$

(iii) Let $\vec{a}_0 = \langle a_{0,i} : i \in I \rangle, \dots, \vec{a}_{n-1} = \langle a_{n-1,i} : i \in I \rangle \in \prod_{i \in I} A_i$. By structural induction we have

$$\begin{aligned} t^{\prod_i A_i}(\vec{a}_0, \dots, \vec{a}_{n-1}) &= \langle t^{\mathbf{A}_i}(a_{0,i}, \dots, a_{n-1,i}) : i \in I \rangle, \quad \text{and,} \\ s^{\prod_i A_i}(\vec{a}_0, \dots, \vec{a}_{n-1}) &= \langle s^{\mathbf{A}_i}(a_{0,i}, \dots, a_{n-1,i}) : i \in I \rangle. \end{aligned}$$

Since $t^{\mathbf{A}_i}(a_{0,i}, \dots, a_{n-1,i}) = s^{\mathbf{A}_i}(a_{0,i}, \dots, a_{n-1,i})$, for all $i \in I$, we get $t^{\prod_i A_i}(\vec{a}_0, \dots, \vec{a}_{n-1}) = s^{\prod_i A_i}(\vec{a}_0, \dots, \vec{a}_{n-1})$. \square

Theorem 3.20. For any class \mathbf{K} of Σ -algebras and any Σ -equation $t \approx s$, if $\mathbf{K} \models t \approx s$, then $\mathbf{HSP}(\mathbf{K}) \models t \approx s$.

Proof. By the lemma, each of the successive entailments implies the following one. $\mathbf{K} \models t \approx s$ implies $\mathbf{P}(\mathbf{K}) \models t \approx s$ implies $\mathbf{SP}(\mathbf{K}) \models t \approx s$ implies $\mathbf{HSP}(\mathbf{K}) \models t \approx s$. \square

The original definition of a free group is metamathematical in nature. Specifically a group is *free* if its generators are *free* in the sense they satisfy no equation that is not satisfied by every choice of elements in every group. In its general form for an arbitrary class \mathbf{K} of Σ -algebras this idea takes the following form.

Definition 3.21. Let \mathbf{K} be a class of Σ -algebra, \mathbf{F} a Σ -algebra, and $Y \subseteq F$. \mathbf{F} is *free over* \mathbf{K} with free generators Y if the following two conditions hold.

- (i) $F = \text{Sg}^{\mathbf{F}}(Y)$.
- (ii) For every Σ -equation $t(x_0, \dots, x_{n-1}) \approx s(x_0, \dots, x_{n-1})$ and any choice of pairwise distinct y_0, \dots, y_{n-1} elements of Y ,

$$t^{\mathbf{F}}(y_0, \dots, y_{n-1}) = s^{\mathbf{F}}(y_0, \dots, y_{n-1}) \quad \text{iff} \quad \mathbf{K} \models t \approx s.$$

For example, the free group over the free generators $y_0, y_1, \dots, y_{\xi}, \dots, \xi < \lambda$, is usually defined to be the set of all *words*, that is finite sequences of variable symbols, $y_{\xi_0}^{k_0} \cdots y_{\xi_{n-1}}^{k_{n-1}}$, where each y_{ξ_i} is different from the symbol on either side of it (i.e., from $y_{\xi_{i-1}}$ and from $y_{\xi_{i+1}}$), k_i is an arbitrary nonzero integer, and $y_{\xi_i}^{k_i}$ denotes the word $\underbrace{y_{\xi_i} y_{\xi_i} \cdots y_{\xi_i}}_{k_i}$ if $k_i > 0$

and the word $\underbrace{y_{\xi_i}^{-1} y_{\xi_i}^{-1} \cdots y_{\xi_i}^{-1}}_{-k_i}$ if $k_i < 0$. The product of the two words $y_{\xi_0}^{k_0} \cdots y_{\xi_{n-1}}^{k_{n-1}}$ and

$y_{\eta_0}^{l_0} \cdots y_{\eta_{n-1}}^{l_{m-1}}$ is defined as follows. Let q be the largest number such that, for all $j < q$, $y_{\xi_{n-j-1}} = y_{\eta_j}$ and $k_{n-j-1} = -l_j$. If $y_{\xi_{n-q-1}} \neq y_{\eta_q}$, then the product is

$$y_{\xi_0}^{k_0} \cdots y_{\xi_{n-q-1}}^{k_{n-q-1}} y_{\eta_q}^{l_q} \cdots y_{\eta_{n-1}}^{l_{m-1}};$$

otherwise, i.e., $y_{\xi_{n-q-1}} = y_{\eta_q}$ but $k_{n-q-1} \neq -l_q$. then the product is

$$y_{\xi_0}^{k_0} \cdots y_{\xi_{n-q-1}}^{k_{n-q-1} + l_q} \cdots y_{\eta_{n-1}}^{l_{m-1}}.$$

For example, the product of $y_0^2 y_1^{-3} y_2^4$ and $y_2^{-4} y_1^3 y_4^{-1}$ is $y_0^2 y_4^{-1}$, and the product of $y_0^2 y_1^{-3} y_2^4$ and $y_2^{-4} y_1^2 y_4^{-1}$ is $y_0^2 y_1^{-1} y_4^{-1}$. The inverse of $y_{\xi_0}^{k_0} \cdots y_{\xi_{n-1}}^{k_{n-1}}$ is $y_{\xi_{n-1}}^{-k_{n-1}} \cdots y_{\xi_0}^{-k_0}$. The identity is the empty word. It is not difficult to show that the $\{\cdot, ^{-1}, e\}$ -algebra constructed in this way is free over the class of groups with free generators $\{y_\xi : \xi < \lambda\}$.

The following lemma ties the universal mapping property to the notion of freedom.

Lemma 3.22. *Let \mathbf{K} be a class of Σ -algebras, and assume that \mathbf{F} is a Σ -algebra such that*

- (i) $\mathbf{F} \in \mathbf{HSP}(\mathbf{K})$, and
- (ii) \mathbf{F} has the universal mapping property over \mathbf{K} wrt a set Y of generators of \mathbf{F} .

Then \mathbf{F} is free over \mathbf{K} with free generators Y .

Proof. Assume (i) and (ii) hold. Then $F = \text{Sg}^{\mathbf{F}}(Y)$ by (ii).

Suppose $t^{\mathbf{F}}(y_0, \dots, y_{n-1}) = s^{\mathbf{F}}(y_0, \dots, y_{n-1})$. Let $\mathbf{A} \in \mathbf{K}$ and $a_0, \dots, a_{n-1} \in A$. Let $h^*: \mathbf{F} \rightarrow \mathbf{A}$ such that $h^*(y_i) = a_i$ for every $i < n$. Then

$$\begin{aligned} t^{\mathbf{A}}(a_0, \dots, a_{n-1}) &= t^{\mathbf{A}}(h^*(y_0), \dots, h^*(y_{n-1})) \\ &= h^*(t^{\mathbf{F}}(y_0, \dots, y_{n-1})) \\ &= h^*(s^{\mathbf{F}}(y_0, \dots, y_{n-1})) \\ &= s^{\mathbf{A}}(h^*(y_0), \dots, h^*(y_{n-1})) \\ &= s^{\mathbf{A}}(a_0, \dots, a_{n-1}). \end{aligned}$$

So $\mathbf{K} \models t \approx s$.

Now assume $\mathbf{K} \models t \approx s$. Then $\mathbf{HSP}(\mathbf{K}) \models t \approx s$ by Thm. 3.20. Thus $\mathbf{F} \models t \approx s$ and hence $t^{\mathbf{F}}(y_0, \dots, y_{n-1}) = s^{\mathbf{F}}(y_0, \dots, y_{n-1})$. \square

The converse of this lemma holds we cannot prove it now.

We want to prove that free algebras over every class exist with any given cardinality λ of free generators. It turns out that the λ -ary clone algebra $\text{Clo}_\lambda(\mathbf{F})$ has this property for any algebra $\mathbf{F} \in \mathbf{HSP}(\mathbf{K})$ with the property that every algebra of \mathbf{K} is a homomorphic image of \mathbf{F} . Clearly the product $\prod_{\mathbf{A} \in \mathbf{K}} \mathbf{A}$ has this property since every algebra in \mathbf{K} is a homomorphic image of it by projection. The problem is that such a product does not exist because it is too big if \mathbf{K} is a proper class. But for \mathbf{F} to serve our purpose we do not need for every algebra of \mathbf{K} to be a homomorphic image of \mathbf{F} , but only the subalgebras of members of \mathbf{K} that can be generated by at most λ generators. Algebras \mathbf{F} with this property do exist. One of them is the following algebra, as we shall see.

$$\prod_{\lambda}^* \mathbf{K} := \prod \{ \text{Te}_\Sigma(X_\lambda) / \alpha : \alpha \in \text{Co}(\text{Te}_\Sigma(X_\lambda)), \text{Te}_\Sigma(X_\lambda) / \alpha \in \mathbf{IS}(\mathbf{K}) \}.$$

Lemma 3.23. *Let \mathbf{K} be a class of Σ -algebras and λ a cardinal.*

- (i) $\prod_{\lambda}^* \mathbf{K} \in \mathbf{PS}(\mathbf{K})$.
- (ii) *For every $\mathbf{A} \in \mathbf{K}$ and every $\mathbf{B} \subseteq \mathbf{A}$ such that \mathbf{B} is generated by at most λ elements we have $\mathbf{B} \preceq \prod_{\lambda}^* \mathbf{K}$.*

Proof. (i) $\prod_{\lambda}^* \mathbf{K} \in \mathbf{P}\{ \mathbf{Te}_{\Sigma}(X_{\lambda})/\alpha : \alpha \in \text{Co}(\mathbf{Te}_{\Sigma}(X_{\lambda})), \mathbf{Te}_{\Sigma}(X_{\lambda})/\alpha \in \mathbf{IS}(\mathbf{K}) \} \subseteq \mathbf{PIS}(\mathbf{K}) = \mathbf{PS}(\mathbf{K})$.

(ii) Let $\mathbf{B} \in \mathbf{S}(\mathbf{K})$ such that \mathbf{B} is generated by at most λ elements. Then by Theorem thm:w6.1(iii) there exists a surjective mapping from the set of variables X_{λ} onto the generators of \mathbf{B} , which extends (uniquely) to an epimorphism h^* from $\mathbf{Te}_{\Sigma}(X_{\lambda})$ onto \mathbf{B} . By the Isomorphism Theorem $h^*_{\alpha}: \mathbf{Te}_{\Sigma}(X_{\lambda})/\alpha \cong \mathbf{B}$, where $\alpha = \text{rker}(h^*)$. So $\mathbf{Te}_{\Sigma}(X_{\lambda})/\alpha \in \mathbf{IS}(\mathbf{K})$, and hence $\mathbf{Te}_{\Sigma}(X_{\lambda})/\alpha$ is one of the factors in the direct product $\prod_{\lambda}^* \mathbf{K}$ and consequently is a homomorphic image of $\prod_{\lambda}^* \mathbf{K}$ under one of the projection mappings. Hence \mathbf{B} is a homomorphic image of $\prod_{\lambda}^* \mathbf{K}$. \square

Definition 3.24. Let \mathbf{K} be a class of Σ -algebras, and let λ be a cardinal. Set

$$\mathbf{Fr}_{\lambda}(\mathbf{K}) := \mathbf{Clo}_{\lambda}(\prod_{\lambda}^* \mathbf{K}).$$

This is called the *free algebra of \mathbf{K} of dimension λ* .

Theorem 3.25. *Let \mathbf{K} be a class of Σ -algebras and λ a cardinal. $\mathbf{Fr}_{\lambda}(\mathbf{K})$ is a free algebra over \mathbf{K} with free generators $X_{\lambda}^{\mathbf{K}} = \{x_{\xi}^{\mathbf{K}} : \xi < \lambda\}$.*

Proof. By Lemma 3.22 it suffices to prove (1) $\mathbf{Fr}_{\lambda}(\mathbf{K}) \in \mathbf{HSP}(\mathbf{K})$, and (2) $\mathbf{Fr}_{\lambda}(\mathbf{K})$ has the universal mapping property over \mathbf{K} wrt $X_{\lambda}^{\mathbf{K}}$.

- (1) By Theorem 3.17 and Lemma 3.23

$$\mathbf{Clo}_{\lambda}(\prod_{\lambda}^* \mathbf{K}) \in \mathbf{SP}(\prod_{\lambda}^* \mathbf{K}) \subseteq \mathbf{SPPS}(\mathbf{K}) \subseteq \mathbf{SSPP}(\mathbf{K}) = \mathbf{SP}(\mathbf{K}) \subseteq \mathbf{HSP}(\mathbf{K}).$$

(2) Let $\mathbf{A} \in \mathbf{K}$ and let $h: X_{\lambda}^{\mathbf{K}} \rightarrow \mathbf{A}$. Let \mathbf{A}' be the subalgebra of \mathbf{A} generated by the image $h(X_{\lambda}^{\mathbf{K}})$. Since \mathbf{A}' is generated by at most λ elements, $\mathbf{A}' \preceq \prod_{\lambda}^* \mathbf{K}$ by Lemma 3.22(ii). So $\mathbf{A}' \in \mathbf{H}(\prod_{\lambda}^* \mathbf{K})$. Hence by Thm. 3.16 $\mathbf{Fr}_{\lambda}(\mathbf{K})$, which, as we recall, is $\mathbf{Clo}_{\lambda}(\prod_{\lambda}^* \mathbf{K})$, has the universal mapping property over \mathbf{A}' wrt $X_{\lambda}^{\mathbf{K}}$. Thus h can be extended to an epimorphism $h^*: \mathbf{Fr}_{\lambda}(\mathbf{K}) \rightarrow \mathbf{A}'$ (it is surjective since \mathbf{A}' is generated by $h(X_{\lambda}^{\mathbf{K}})$), and hence, again identifying h^* with its graph, $h^*: \mathbf{Fr}_{\lambda}(\mathbf{K}) \rightarrow \mathbf{A}$ and $h^* \upharpoonright X_{\lambda}^{\mathbf{K}}$. So $\mathbf{Fr}_{\lambda}(\mathbf{K})$ has the UMP over \mathbf{K} wrt $X_{\lambda}^{\mathbf{K}}$. \square