

By Lemma 2.6  $\text{Te}_\Sigma(X)$  has the following recursive characterization.  $X \subseteq \text{Te}_\Sigma(X)$  (base step). If  $t_1, \dots, t_n \in \text{Te}_\Sigma(X)$ , then for every  $\sigma \in \Sigma_n$ ,

$$\sigma^{\text{Te}_\Sigma(X)}(t_1, \dots, t_n) = \sigma \frown t_1 \frown \dots \frown t_n \in \text{Te}_\Sigma(X),$$

where “ $\frown$ ” denotes the concatenation of strings. In the sequel we will normally omit the superscript on  $\sigma^{\text{Te}_\Sigma(X)}$  and write simply  $\sigma(t_1, \dots, t_n)$  for  $\sigma^{\text{Te}_\Sigma(X)}(t_1, \dots, t_n)$ ; we leave it to context to indicate that we are applying the term-builder operation of the term algebra.

The set of new terms that are created at the the  $n$ -th step in this process is  $E_n(X) \setminus E_{n-1}(X)$ .  $n$  is called their *structural height*. Structural induction can be viewed as induction on the structural height.

Example. Let  $\Sigma = \{\vee, \wedge, \neg, 0, 1\}$ , the signature of Boolean algebras, and let

$$t = \wedge - x_1 0 \wedge x_2 1.$$

Here is the “parse tree” for  $t$ . (We use quotes in refereeing to the parse tree because we do not give a mathematically precise definition. But the intuition is clear and proves useful.)

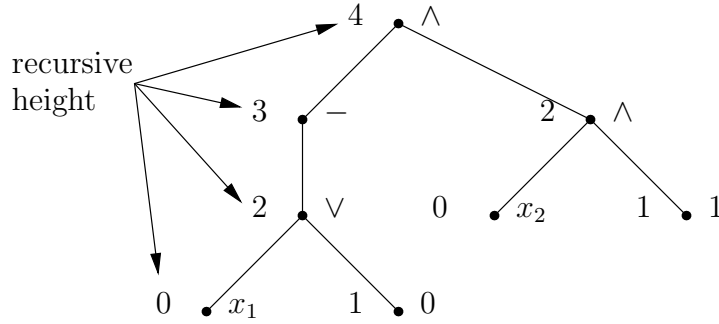


FIGURE 22

If a  $\Sigma$ -term contains no nullary operations its structural height is the same as the height of its parse tree.

Although technically not needed we will use parentheses to make terms easy to parse, We also write binary operations in between their arguments, as is ordinarily done, rather than to the left of them. If we do this for every binary operation, parentheses are necessary for unique parsing. For example, we will write the above term  $t$  in the form  $-(x_1 \vee 0) \wedge (x_2 \wedge 1)$ .

**Theorem 3.5.**  $\text{Te}_\Sigma(X)$  has the *unique parsing property wrt*  $X$ , i.e.,

- (i)  $x \neq \sigma(t_1, \dots, t_n)$  for every  $x \in X$ , every  $\sigma \in \Sigma_n$ , and all  $t_1, \dots, t_n \in \text{Te}_\Sigma(X)$ .
- (ii) For all  $n, m \in \omega$ , every  $\sigma \in \Sigma_n$ , every  $\tau \in \Sigma_m$ , and all  $t_1, \dots, t_n, s_1, \dots, s_m \in \text{Te}_\Sigma(X)$ ,

$$\sigma(t_1, \dots, t_n) = \tau(s_1, \dots, s_m) \text{ iff } n = m, \sigma = \tau, \text{ and } t_i = s_i \text{ for all } i \leq n.$$

The proof of this theorem is left as an exercise.

The theorem says that the parse tree for each  $\Sigma$ -term is unique.

**Corollary 3.6.**  $\text{Te}_\Sigma(X)$  has the *unique mapping property over*  $\text{Alg}(\Sigma)$  wrt  $X$ .

*Proof.* Let  $\mathbf{A}$  be any  $\Sigma$ -algebra and let  $h: X \rightarrow A$  be any map. Define  $h^*: \text{Te}_\Sigma(X) \rightarrow A$  by structural induction, i.e., induction on the structural height of a  $\Sigma$ -term  $t$ . If  $t = x$ ,  $h^*(t) = h(x)$ . If  $t = \sigma(t_1, \dots, t_n)$ ,  $h^*(t) = \sigma^{\mathbf{A}}(h^*(t_1), \dots, h^*(t_n))$ .  $h^*$  is well defined by the unique parsing property. It is clearly a homomorphism that agrees with  $h$  on  $X$ .  $\square$

A mapping  $h: X \rightarrow A$  is called an *assignment* of elements of  $A$  to the variables in  $X$ , and its unique extension  $h^*$  to a homomorphism from  $\mathbf{Te}_\Sigma(X)$  to  $\mathbf{A}$  is called the *evaluation map* based on this assignment.

Let  $n \in \omega$  be fixed but arbitrary and let  $x_1, \dots, x_n$  be a sequence of distinct variable symbols. Let  $X = \{x_1, \dots, x_n\}$ . We often write a term  $t$  in the form  $t(x_1, \dots, x_n)$  as an alternative way of specifying that the variables of  $t$  are in  $X$ , i.e., that  $t \in \text{Te}_\Sigma(X)$ . In the same spirit we write  $t^{\mathbf{A}}(a_1, \dots, a_n)$  for the image  $h^*(t)$  of  $t$  in  $\mathbf{A}$  under the evaluation map based on the assignment  $h(x_i) = a_i$  for all  $i \leq n$ .

For example, let  $t(x_1, x_2)$  be the term  $-(x_1 \vee 0) \wedge (x_2 \wedge 1)$  considered above, and let  $\mathbf{B}_2 = \langle \{0, 1\}, \vee^{\mathbf{B}_2}, \wedge^{\mathbf{B}_2}, -^{\mathbf{B}_2}, 0, 1 \rangle$  be the 2-element Boolean algebra. Then

$$\begin{aligned} t_2^{\mathbf{B}_2}(0, 1) &= h^*(t) = h^*(-(x_1 \vee 0) \wedge (x_2 \wedge 1)) \\ &= -^{\mathbf{B}_2}(h(x_1) \vee^{\mathbf{B}_2} 0) \wedge^{\mathbf{B}_2} (h(x_2) \wedge^{\mathbf{B}_2} 1) \\ &= -^{\mathbf{B}_2}(0 \vee^{\mathbf{B}_2} 0) \wedge^{\mathbf{B}_2} (1 \wedge^{\mathbf{B}_2} 1) \\ &= 0. \end{aligned}$$

Terms in  $\text{Te}_\Sigma(\{x_1, \dots, x_n\})$  are said to be *n-ary*. An *n-ary* term  $t$  determines, for each  $\Sigma$ -algebra  $\mathbf{A}$ , an *n-ary* operation  $t^{\mathbf{A}}: A^n \rightarrow A$  on  $A$  that maps  $\langle a_1, \dots, a_n \rangle$  to  $t^{\mathbf{A}}(a_1, \dots, a_n)$ .

**Definition 3.7.** Let  $\mathbf{A}$  be a  $\Sigma$ -algebra. An *n-ary* operation  $f: A^n \rightarrow A$  on  $A$  is called a *term* or *derived operation* of  $\mathbf{A}$  if there exists a  $\Sigma$ -term  $t(x_1, \dots, x_n)$  such that  $f = t^{\mathbf{A}}$ .

The set of all term operations of  $\mathbf{A}$  of rank  $n$  is denoted by  $\text{Clo}_n(\mathbf{A})$  and is called the *n-ary clone* of  $\mathbf{A}$ .

By the *clone* of  $\mathbf{A}$ , in symbols,  $\text{Clo}_\omega(\mathbf{A})$ , we mean the union of the *n-ary* clones over all  $n \in \omega$ , i.e.,  $\bigcup_{n \in \omega} \text{Clo}_n(\mathbf{A})$ . Some authors define the clone of  $\mathbf{A}$  to be the  $\omega$ -sorted set  $\langle \text{Clo}_n(\mathbf{A}) : n \in \omega \rangle$ .

Here are some basic facts about term operations; all have easy proofs by structural induction.

(1) Subuniverses are closed under term operations, i.e., if  $B \in \text{Sub}(\mathbf{A})$  and  $t(x_1, \dots, x_n)$  is a  $\Sigma$ -term, then for all  $b_1, \dots, b_n \in B$ ,  $t^{\mathbf{A}}(b_1, \dots, b_n) \in B$ . Moreover, if  $\mathbf{B} \subseteq \mathbf{A}$ , then, for all  $b_1, \dots, b_n \in B$ ,  $t^{\mathbf{B}}(b_1, \dots, b_n) = t^{\mathbf{A}}(a_1, \dots, a_n)$ , i.e.,  $t^{\mathbf{B}} = t^{\mathbf{A}} \upharpoonright B^n$ .

(2) Homomorphisms preserve term operations, i.e., if  $h: \mathbf{A} \rightarrow \mathbf{B}$ , then for all  $a_1, \dots, a_n \in A$ ,  $h(t^{\mathbf{A}}(a_1, \dots, a_n)) = t^{\mathbf{B}}(h(a_1), \dots, h(a_n))$ .

(3) Every term operation has the substitution property wrt congruences, i.e., if  $\alpha \in \text{Co}(\mathbf{A})$  and  $a_1 \alpha b_1, \dots, a_n \alpha b_n$ , then  $t^{\mathbf{A}}(a_1, \dots, a_n) \alpha t^{\mathbf{A}}(b_1, \dots, b_n)$ .

**Definition 3.8.** Let  $\Sigma, \Sigma'$  be signatures, and let  $\mathbf{A}, \mathbf{A}'$  be  $\Sigma$ - and  $\Sigma'$ -algebras, respectively.  $\mathbf{A}'$  is *termwise definable* in  $\mathbf{A}$  if every fundamental operation of  $\mathbf{A}'$  is a term operation of  $\mathbf{A}$ .  $\mathbf{A}$  and  $\mathbf{A}'$  are *termwise definitionally equivalent* if each is termwise definable in the other.

It follows immediately from the basic facts about term operations listed above that, if  $\mathbf{A}'$  is termwise definable in  $\mathbf{A}$  then  $\text{Sub}(\mathbf{A}) \subseteq \text{Sub}(\mathbf{A}')$  and  $\text{Co}(\mathbf{A}) \subseteq \text{Co}(\mathbf{A}')$ . Thus termwise definitionally equivalent algebras have the same subuniverses and same congruence relations. Moreover if  $\mathbf{B}$  and  $\mathbf{B}'$  are other termwise definitionally equivalent  $\Sigma$ - and  $\Sigma'$ -algebras, respectively, then  $\text{Hom}(\mathbf{A}', \mathbf{B}') = \text{Hom}(\mathbf{A}, \mathbf{B})$ .

**Lemma 3.9.**  *$\mathbf{A}$  and  $\mathbf{A}'$  are termwise definitionally equivalent iff  $\text{Clo}_\omega(\mathbf{A}) = \text{Clo}_\omega(\mathbf{A}')$ , i.e.,  $\text{Clo}_n(\mathbf{A}) = \text{Clo}_n(\mathbf{A}')$  for every  $n \in \omega$ .*

The proof is left as an exercise.

Termwise definitionally equivalent algebras have essentially the same algebraic properties and are often identified. From this point of view, what really defines an algebra is its clone, the set of all its term operations. Its fundamental operations in this view are useful mainly for conveniently specifying, i.e., generating, the clone, and there are usually many different choices of the fundamental operations that can serve this purpose. This is part of the motivation behind the categorical approach to universal algebra. This is illustrated in the following example.

Example. Let  $\mathbf{B} = \langle \mathbf{B}, \vee, \wedge, -, 0, 1 \rangle$  be a Boolean algebra over the signature  $\Sigma = \{\vee, \wedge, -, 0, 1\}$ , and consider its two reducts  $\mathbf{A} = \langle \mathbf{A}, \vee, -, 0, 1 \rangle$  and  $\mathbf{A}' = \langle \mathbf{A}, \wedge, -, 0, 1 \rangle$  to the respective signatures  $\Sigma = \{\vee, -, 0, 1\}$  and  $\Sigma' = \{\wedge, -, 0, 1\}$ . By DeMorgan's laws and the law of double negation we know that the identities  $x_1 \wedge x_2 \approx -(-x_1 \vee -x_2)$  and  $x_1 \vee x_2 \approx -(-x_1 \wedge -x_2)$  are satisfied in  $\mathbf{B}$ . Thus  $\mathbf{A}$  and  $\mathbf{A}'$  are termwise definitionally equivalent.

We note that the basic facts about term operations can often be used to show that and operation on an algebra is not a term operation. Consider for example the semigroup reduct  $\langle \mathbb{Z}, + \rangle$  of the group  $\mathbb{Z} = \langle \mathbb{Z}, +, -, 0 \rangle$  of integers.  $-$  is not a term operation of  $\langle \mathbb{Z}, + \rangle$  because, for example, the subuniverse  $\omega$  of  $\langle \mathbb{Z}, + \rangle$  is not closed under  $-$ . Thus  $\langle \mathbb{Z}, + \rangle$  and  $\mathbb{Z}$  are not termwise definitionally equivalent. This is the reason that, from the point of view of universal algebras, groups of type I and type II (See the examples following Def. 2.2) are considered different.

The best known example of an  $n$ -ary clone over an algebra is the set of polynomial functions in  $n$  indeterminants over a ring (in particular the ring of real numbers) with integer coefficients. Let  $\mathbb{R} = \langle \mathbb{R}, +, \cdot, -, 0, 1 \rangle$  be the ring of real numbers (a field). Then

$$\text{Clo}_1(\mathbb{R}) = \{ (k_0 + k_1x + \cdots + k_nx^n)^{\mathbb{R}} : k_0, \dots, k_n \in \mathbb{Z} \}.$$

We make two observations:

(1) An element of this clone is a polynomial *function* over  $\mathbb{R}$ , as opposed to a formal polynomial; this distinction is usually made in a course in abstract algebra, but not in the calculus where polynomials are always thought of as functions. Of course formal polynomials correspond to our terms.

(2) The elements of the clone are the polynomials with integer coefficients. For example, a typical example would be the term operation determined by term  $3 + (-2)x + 3x^2 = 1 + 1 + 1 + -x + -x + x \cdot x + x \cdot x + x \cdot x$ . In the construction of the term we can use only the fundamental operation symbols of the signature  $\{+, \cdot, -, 0, 1\}$ . However, we can get all

the polynomial functions simply by expanding the signature to include a symbol for each element of  $\mathbb{R}$ . Here is how it is done in general.

Let  $\Sigma$  be an arbitrary signature and let  $\mathbf{A}$  be a  $\Sigma$ -algebra. For each  $a \in A$  let  $\bar{a}$  be a new nullary operation symbol. Let  $\Sigma_A = \Sigma \cup \{\bar{a} : a \in A\}$  and let  $\mathbf{A}^+$  be the  $\Sigma_A$ -algebra

$$\mathbf{A}^+ = \langle A, \langle \sigma^{\mathbf{A}} : \sigma \in \Sigma \rangle \cup \langle \bar{a}^{\mathbf{A}} : a \in A \rangle \rangle,$$

where  $\bar{a}^{\mathbf{A}} = a$  for every  $a \in A$ .

**Definition 3.10.** Let  $n \in \omega$ . A element of  $\text{Clo}_n(\mathbf{A}^+)$  is called an *n-ary polynomial operations over  $\mathbf{A}$* .

Notice that the nullary fundamental operation  $\bar{a}^{\text{Clo}_n(\mathbf{A}^+)}$  of  $\mathbf{Clo}_n(\mathbf{A}^+)$  denotes the constant  $n$ -ary operation on  $A$  with (unique) value  $a$ , i.e.,  $\bar{a}^{\text{Clo}_n(\mathbf{A}^+)}(b_1, \dots, b_n) = a$  for all  $\langle b_1, \dots, b_n \rangle \in A^n$ .

For every natural number  $n$ ,  $\text{Clo}_n(\mathbb{R}^+)$  is the set of all polynomial functions over the real numbers in  $n$ -indeterminants, and  $\text{Clo}_n(\mathbb{R})$  is subset consisting of polynomial functions with only integer coefficients.

An important and well known feature of both the term operations and the the polynomial operations over the reals is that they can both have a ring structure. We now show that a similar situation holds for the term and polynomial operations over any algebra and any signature. For this purpose we need the following lemma.

**Lemma 3.11.** Let  $t_1(x_1, \dots, x_n), \dots, t_k(x_1, \dots, x_n)$  be  $n$ -ary  $\Sigma$ -terms. Let  $s(x_1, \dots, x_n) = \sigma(t_1, \dots, t_k)$  for some  $\sigma \in \Sigma_k$ . Then for every  $\Sigma$ -algebra  $\mathbf{A}$  and all  $a_1, \dots, a_k \in A$

$$s^{\mathbf{A}}(a_1, \dots, a_n) = \sigma^{\mathbf{A}}(t^{\mathbf{A}}(a_1, \dots, a_k), \dots, t^{\mathbf{A}}(a_1, \dots, a_k)).$$

*Proof.* Let  $X = \{x_1, \dots, x_n\}$  and let  $h: X \rightarrow A$  such that  $h(x_i) = a_i$  for every  $i \leq n$ . Let  $h^*: \mathbf{Te}_\Sigma(X) \rightarrow \mathbf{A}$  such that  $h^* \upharpoonright X = h$ . Then we have

$$\begin{aligned} s^{\mathbf{A}}(a_1, \dots, a_n) &= h^*(s), && \text{by definition of } s^{\mathbf{A}} \\ &= h^*(\sigma(t_1, \dots, t_m)) \\ &= \sigma^{\mathbf{A}}(h^*(t_1), \dots, h^*(t_k)), && \text{since } h^* \text{ is a homomorphism} \\ &= \sigma^{\mathbf{A}}(t_1^{\mathbf{A}}(a_1, \dots, a_n), \dots, t_k^{\mathbf{A}}(a_1, \dots, a_n)). \end{aligned}$$

□

Let  $\Sigma$  be a signature and  $\mathbf{A}$  a  $\Sigma$ -algebra. For each  $n \in \omega$  we can give the set  $\text{Clo}_n(\mathbf{A})$  of  $n$ -ary term operations of  $\mathbf{A}$  the structure of a  $\Sigma$ -algebra  $\mathbf{Clo}_n(\mathbf{A})$ . Let  $\sigma \in \Sigma_k$  and let  $f_1, \dots, f_k \in \text{Clo}_n(\mathbf{A})$ . Choose  $n$ -ary terms  $t_1(x_1, \dots, x_n), \dots, t_k(x_1, \dots, x_n)$  such that  $t_i^{\mathbf{A}} = f_i$  for each  $i \leq k$ . Define  $\sigma^{\text{Clo}_n(\mathbf{A})}(f_1, \dots, f_k) = t^{\mathbf{A}}$ , where  $t = \sigma(t_1, \dots, t_n)$ . Since the definition depends on a choice of representative terms for the term operations  $f_1, \dots, f_k$  we have to show  $\sigma^{\text{Clo}_n(\mathbf{A})}$  is well defined, i.e., it does not depend on the particular choice of representative terms.

Suppose  $f_1 = s_1^{\mathbf{A}}, \dots, f_k = s_k^{\mathbf{A}}$  and  $s = \sigma(s_1, \dots, s_k)$ . Then for all  $a_1, \dots, a_n \in A$  we have

$$\begin{aligned}
 t^{\mathbf{A}}(a_1, \dots, a_n) &= (\sigma(t_1, \dots, t_k))^{\mathbf{A}}(a_1, \dots, a_n) \\
 &= \sigma^{\mathbf{A}}(t_1^{\mathbf{A}}(a_1, \dots, a_n), \dots, t_k^{\mathbf{A}}(a_1, \dots, a_n)), \quad \text{by Lemma 3.11} \\
 &= \sigma^{\mathbf{A}}(f_1(a_1, \dots, a_n), \dots, f_k(a_1, \dots, a_n)) \\
 &= \sigma^{\mathbf{A}}(s_1^{\mathbf{A}}(a_1, \dots, a_n), \dots, s_k^{\mathbf{A}}(a_1, \dots, a_n)) \\
 &= (\sigma(s_1, \dots, s_k))^{\mathbf{A}}(a_1, \dots, a_n), \quad \text{by Lemma 3.11} \\
 &= s^{\mathbf{A}}(a_1, \dots, a_n).
 \end{aligned}$$

So  $\sigma^{\mathbf{Clo}_n(\mathbf{A})}$  is well defined.

**Definition 3.12.** let  $\Sigma$  be a  $\Sigma$ -algebra and  $n \in \omega$ . Then

- (i)  $\mathbf{Clo}_n(\mathbf{A}) = \langle \mathbf{Clo}_n(\mathbf{A}), \sigma^{\mathbf{Clo}_n(\mathbf{A})} \rangle_{\sigma \in \Sigma}$ . It is called the *n-ary clone algebra over  $\mathbf{A}$* , or alternatively, the *algebra of n-ary term operations over  $\mathbf{A}$* .
- (ii) The  $\Sigma_{\mathbf{A}}$ -algebra  $\mathbf{Clo}_n(\mathbf{A}^+)$  is called the *algebra of n-ary polynomial operations over  $\mathbf{A}$* .

We make several observations.

- (1)  $\mathbf{Clo}_n(\mathbf{A})$  is generated by  $\{x_1^{\mathbf{A}}, \dots, x_n^{\mathbf{A}}\}$  (exercise).
- (2) If  $\mathbf{A}$  is nontrivial (i.e.,  $|A| \geq 2$ ), then, for all  $i, j \leq n$ ,

$$i \neq j \text{ implies } x_i^{\mathbf{A}} \neq x_j^{\mathbf{A}},$$

i.e.,  $|\{x_1^{\mathbf{A}}, \dots, x_n^{\mathbf{A}}\}| = n$ . To see this consider any two distinct elements  $a$  and  $b$  of  $\mathbf{A}$ . Then

$$x_i^{\mathbf{A}}(a, \dots, a, b, a, \dots, a) = a \quad \text{and} \quad x_j^{\mathbf{A}}(a, \dots, a, b, a, \dots, a) = b.$$

(3) The mapping  $a \mapsto \bar{a}^{\mathbf{Clo}_n(\mathbf{A}^+)}$  is a monomorphism from  $\mathbf{A}^+$  into  $\mathbf{Clo}_n(\mathbf{A}^+)$ . This is an easy exercise. Thus if we identify  $a$  and  $\bar{a}^{\mathbf{Clo}_n(\mathbf{A}^+)}$  we can assume the  $\mathbf{A}^+$  is a subalgebra of  $\mathbf{Clo}_n(\mathbf{A}^+)$ .

We all know that the ring of integer-coefficient polynomials over  $\mathbb{R}$  has the universal mapping property over the  $\mathbb{R}$  wrt to the set of indeterminants. This holds in general as we now show.

**Theorem 3.13.** *let  $\mathbf{A}$  be a  $\Sigma$ -algebra and  $n \in \omega$ . Then  $\mathbf{Clo}_n(\mathbf{A})$  has the universal mapping property over  $\mathbf{A}$  (i.e., over  $\{\mathbf{A}\}$ ) wrt to  $\{x_1^{\mathbf{A}}, \dots, x_n^{\mathbf{A}}\}$ .*

*Proof.* If  $\mathbf{A}$  is trivial, then the theorem is trivially true. Assume  $\mathbf{A}$  is nontrivial. Let  $h: \{x_1^{\mathbf{A}}, \dots, x_n^{\mathbf{A}}\} \rightarrow A$ . Define  $h^*: \mathbf{Clo}_n(\mathbf{A}) \rightarrow A$  by

$$h^*(t^{\mathbf{A}}) = t^{\mathbf{A}}(h(x_1^{\mathbf{A}}), \dots, h(x_n^{\mathbf{A}})), \quad \text{for every } n\text{-ary term } t(x_1, \dots, x_n).$$

$h^*$  is obviously well defined. We check it is a homomorphism.

$$\begin{aligned}
h^*(\sigma^{\mathbf{Clo}_n(\mathbf{A})}(t_1^{\mathbf{A}}, \dots, t_k^{\mathbf{A}})) & \\
&= h^*((\sigma(t_1, \dots, t_n))^{\mathbf{A}}), && \text{by defn. of } \sigma^{\mathbf{Clo}_n(\mathbf{A})} \\
&= \sigma(t_1, \dots, t_n)^{\mathbf{A}}(h(x_1^{\mathbf{A}}), \dots, h(x_n^{\mathbf{A}})), && \text{by defn. of } h^* \\
&= \sigma^{\mathbf{A}}(t_1^{\mathbf{A}}(h(x_1^{\mathbf{A}}), \dots, h(x_n^{\mathbf{A}})), \dots, t_k^{\mathbf{A}}(h(x_1), \dots, h(x_n))), && \text{by Lemma 3.11} \\
&= \sigma^{\mathbf{A}}(h^*(t_1^{\mathbf{A}}), \dots, h^*(t_k^{\mathbf{A}})), && \text{by defn. of } h^*.
\end{aligned}$$

Finally, we note that, for each  $i \leq n$ ,  $h^*(x_i^{\mathbf{A}}) = x_i^{\mathbf{A}}(h(x_1^{\mathbf{A}}), \dots, h(x_n^{\mathbf{A}})) = h(x_i^{\mathbf{A}})$ .  $\square$

**Corollary 3.14.** *For every  $h: \{x_1^{\mathbf{A}^+}, \dots, x_n^{\mathbf{A}^+}\} \rightarrow A$  there is a unique homomorphism  $h^*: \mathbf{Clo}_n(\mathbf{A}^+) \rightarrow \mathbf{A}^+$  such that*

- (i)  $h^* \upharpoonright \{x_1^{\mathbf{A}^+}, \dots, x_n^{\mathbf{A}^+}\} = h$ .
- (ii)  $h^* \upharpoonright A = \Delta_A$ .

*Proof.* By the theorem there is a unique  $h^*: \mathbf{Clo}_n(\mathbf{A}^+) \rightarrow \mathbf{A}^+$  such that condition (i) holds. Since  $h^*$  is a homomorphism over the extended signature  $\Sigma_{\mathbf{A}}$ ,  $h^*$  is identity on  $A$  since for each  $a \in A$ ,  $h^*(a) = h^*(\bar{a}^{\mathbf{A}}) = \bar{a}^{\mathbf{Clo}_n(\mathbf{A}^+)} = a$ ; the last equality holds because we have identified  $\bar{a}^{\mathbf{Clo}_n(\mathbf{A}^+)}$  with  $a$ .  $\square$

The property of the  $n$ -ary clone algebra over  $\mathbf{A}$  that sets it apart from the  $n$ -ary term algebra is that it is a subdirect power of  $\mathbf{A}$ , as we show in the next theorem.

**Theorem 3.15.** *Let  $\mathbf{A}$  be a  $\Sigma$ -algebra and  $n \in \omega$ . Then  $\mathbf{Clo}_n(\mathbf{A}) \in \mathbf{SP}(\mathbf{A})$ . In fact,  $\mathbf{Clo}_n(\mathbf{A})$  is the subalgebra of  $\mathbf{A}^{A^n}$  generated by  $\{\pi_1, \dots, \pi_n\}$ , where  $\pi_i: A^{A^n}$  is the  $i$ -projection function.*

*Proof.* Let  $X = \{x_1, \dots, x_n\}$  and let  $h^*: \mathbf{Te}_{\Sigma}(X) \rightarrow \mathbf{A}^{A^n}$  such that  $h^*(x_i) = \pi_i$  for each  $i \leq n$ . By Thm. 2.14(iii),  $h^*(\mathbf{Te}_{\Sigma}(X)) = \mathbf{Sg}^{\mathbf{A}^{A^n}}(\{\pi_1, \dots, \pi_n\})$ .

The claim is that, for every  $t \in \mathbf{Te}_{\Sigma}(X)$ ,  $h^*(t) = t^{\mathbf{A}}$ , and hence

$$\mathbf{Clo}_n(\mathbf{A}) = h^*(\mathbf{Te}_{\Sigma}(X)) = \mathbf{Sg}^{\mathbf{A}^{A^n}}(\{\pi_1, \dots, \pi_n\}).$$

This is proved by induction on the structural height of  $t$ . For every  $\langle a_1, \dots, a_n \rangle \in A^n$ ,  $h^*(x_i)(a_1, \dots, a_n) = \pi_i(a_1, \dots, a_n) = a_i = x_i^{\mathbf{A}}(a_1, \dots, a_n)$ . So  $h^*(x_i) = x_i^{\mathbf{A}}$  for every  $i \leq n$ .

$$\begin{aligned}
&h^*(\sigma(t_1, \dots, t_k))(a_1, \dots, a_n) \\
&= \sigma^{\mathbf{A}^{A^n}}(h^*(t_1), \dots, h^*(t_k))(a_1, \dots, a_n), && \text{since } h^* \text{ is a homomorphism} \\
&= \sigma^{\mathbf{A}}(h^*(t_1)(a_1, \dots, a_n), \dots, h^*(t_k)(a_1, \dots, a_n)), && \text{by defn. of } \sigma^{\mathbf{A}^{A^n}} \\
&= \sigma^{\mathbf{A}}(t_1^{\mathbf{A}}(a_1, \dots, a_n), \dots, t_k^{\mathbf{A}}(a_1, \dots, a_n)), && \text{by ind. hyp.} \\
&= (\sigma(t_1, \dots, t_k))^{\mathbf{A}}(a_1, \dots, a_n), && \text{by Lemma 3.11.}
\end{aligned}$$

Since this holds for every  $\langle a_1, \dots, a_n \rangle \in A^n$ , we get  $h^*(\sigma(t_1, \dots, t_n)) = (\sigma(t_1, \dots, t_n))^{\mathbf{A}}$ . This proves the claim and hence the theorem.  $\square$

It is easy to see that, whenever  $n \leq m$ , each  $n$ -ary term  $t(x_1, \dots, x_n)$  is also an  $m$ -ary term  $t(x_1, \dots, x_m)$ , even though the variables  $x_{n+1}, \dots, x_m$  don't actually occur in  $t$ . Functionally, if we write  $f$  for  $t^{\mathbf{A}}$  as a function form  $A^n$  to  $A$ , and write  $g$  for  $t^{\mathbf{A}}$  as a function from  $A^m$  to  $A$ , then, for every  $\langle a_1, \dots, a_m \rangle \in A^m$ ,  $g(a_1, \dots, a_m) = f(a_1, \dots, a_n)$ . It is easy to check that the mapping  $f \mapsto g$  gives an embedding of  $\mathbf{Clo}_n(\mathbf{A})$  in  $\mathbf{Clo}_m(\mathbf{A})$ . We identify  $\mathbf{Clo}_n(\mathbf{A})$  with its image in  $\mathbf{Clo}_m(\mathbf{A})$  so that  $\mathbf{Clo}_n(\mathbf{A})$  can actually be viewed as a subalgebra of  $\mathbf{Clo}_m(\mathbf{A})$ . The whole clone  $\mathbf{Clo}_\omega(\mathbf{A}) = \bigcup_{n \in \omega} \mathbf{Clo}_n(\mathbf{A})$  can then be made into a  $\Sigma$ -algebra in the natural way: if  $t(x_1, \dots, x_{n_i})$  is an  $n_i$ -ary term for  $i \leq k$  and  $\sigma \in \Sigma_k$ , then  $\sigma(t_1, \dots, t_k)$  is an  $m$ -ary term where  $m = \text{Max}\{n_1, \dots, n_k\}$  and thus  $\sigma^{\mathbf{Clo}_\omega(\text{sb}\mathbf{A})}(t_1^{\mathbf{A}}, \dots, t_k^{\mathbf{A}}) = \sigma^{\mathbf{Clo}_m(\mathbf{A})}(t_1^{\mathbf{A}}, \dots, t_k^{\mathbf{A}})$ .

$\mathbf{Clo}_\omega(\mathbf{A})$  can be thought of as the set of term operations associated with terms over the infinite set of variables  $X = \{x_1, x_2, x_3, \dots\}$ . Each such term contains only a finite number of variables, but of course the number of different variables that can occur in a term is unbounded. Thus  $t^{\mathbf{A}}: A^\omega \rightarrow A$  is an operation with a nominally infinite number of arguments, but it is independent of all but an infinite number of them.

It is not difficult to check that  $\mathbf{Clo}_\omega(\mathbf{A})$  has the UMP over  $\mathbf{A}$  wrt to the infinite set  $\{x_1^{\mathbf{A}}, x_2^{\mathbf{A}}, x_3^{\mathbf{A}}, \dots\}$ , and that  $\mathbf{Clo}_\omega(\mathbf{A}) \cong ; \subseteq \mathbf{A}^{A^\omega}$ . But we want to show how to construct algebras of term operations over sets of variables of arbitrary large cardinality which will give us algebras with the UMP wrt arbitrary large sets. This requires a little transfinite cardinal arithmetic.

In standard set theory a cardinal number is identified with the set of all ordinal numbers less than it, i.e., for each cardinal number  $\alpha$ ,  $\alpha = \{\xi : \xi \text{ is an ordinal and } \xi < \alpha\}$ . Consider for example the finite ordinals:  $0 < 1 < 2 < 3 < \dots$ . Each of them is also a cardinal number (finite ordinals and cardinals are the same).  $0 = \emptyset$ ,  $1 = \{0\} = \{\emptyset\}$ ,  $2 = \{0, 1\} = \{\emptyset, \{\emptyset\}\}$ ,  $\dots$ ,  $n = \{0, 1, 2, \dots, n-1\}$ . The first infinite cardinal is  $\aleph_0 = \omega = \{0, 1, 2, 3, \dots\}$ . The next infinite cardinal is  $\aleph_1 = \{0, 1, 2, \dots, \omega, \omega+1, \omega+2, \dots, \omega+\omega, \omega+\omega+1, \dots\}$ .<sup>2</sup> The “+” in “ $\omega+1$ ,” “ $\omega+2$ ,” “ $\dots$ ,” “ $\omega+\omega+1$ ,” “ $\dots$ ” is ordinal addition, but don't worry about it, it doesn't play any role in our work. Notice that for any cardinal  $\alpha$  and any ordinal  $\xi$ ,  $\xi \in \alpha$  iff  $\xi < \alpha$ ; in particular,  $n \in \omega$  iff  $n < \omega$ .

Let  $\alpha$  be an infinite cardinal, and let  $X_\alpha = \{x_\xi : \xi \in \alpha\}$  be a set of pairwise distinct variable symbols indexed by the ordinals less than  $\alpha$  so that the cardinality of  $X$  is  $\alpha$ . For every  $t \in \text{Te}_\Sigma(X)$  we define a term operation  $t^{\mathbf{A}}: A^\alpha \rightarrow A$  just as before by replacing the finite cardinal “ $n$ ” everywhere by “ $\alpha$ ”. Note that an  $\alpha$ -ary term  $t(\langle x_\xi : \xi < \alpha \rangle)$  still contains occurrences only a finite number of the  $x_\xi$ , and the corresponding term operation  $t^{\mathbf{A}}$  is independent of all but a finite number of its  $\alpha$  arguments. The set of all  $\alpha$ -ary term operations is called the  $\alpha$ -ary clone of  $\mathbf{A}$  and is denoted by  $\mathbf{Clo}_\alpha(\mathbf{A})$ .  $\mathbf{Clo}_\alpha(\mathbf{A})$  can be given the structure of a  $\Sigma$ -algebra  $\mathbf{Clo}_\alpha(\mathbf{A})$ , just like  $\mathbf{Clo}_n(\mathbf{A})$  for finite  $n$ . The following two theorems are proved just like Theorems 3.13 and 3.15, the corresponding theorems for finitary clone algebras, essentially by replacing “ $n$ ” everywhere by “ $\alpha$ ”.

**Theorem 3.16.** *let  $\mathbf{A}$  be a  $\Sigma$ -algebra and  $\alpha$  any infinite cardinal Then  $\mathbf{Clo}_\alpha(\mathbf{A})$  has the universal mapping property over  $\mathbf{A}$  wrt to  $\{x_\xi^{\mathbf{A}} : \xi < \alpha\}$ .*

<sup>2</sup>By the Continuum Hypothesis  $\aleph_1$  is the cardinality of the continuum, i.e.,  $\aleph_1 = |\mathbb{R}|$ . But the Continuum Hypothesis is not provable from the axioms of set theory, so there are models of set theory in which  $\aleph_1$  is much smaller than  $|\mathbb{R}|$ .

**Theorem 3.17.** *Let  $\mathbf{A}$  be a  $\Sigma$ -algebra and  $\alpha$  any infinite cardinal. Then  $\mathbf{Clo}_\alpha(\mathbf{A}) \in \mathbf{SP}(\mathbf{A})$ . In fact,  $\mathbf{Clo}_\alpha(\mathbf{A})$  is the subalgebra of  $\mathbf{A}^{A^\alpha}$  generated by  $\{\pi_\xi : \xi \in \alpha\}$ , where  $\pi_\xi: A^{A^\alpha}$  is the  $\xi$ -projection function.*