

**Corollary 2.72.** *Let  $\mathbf{A}$  be a nontrivial  $\Sigma$ -algebra. Then  $\mathbf{A}$  is subdirectly irreducible (SDI) iff  $\mathbf{A}$  is subdirectly embedding irreducible (SDEI).*

*Proof.* We have already observed that SDEI implies SDI trivially. For the implication in the other direction suppose  $\mathbf{A}$  is SDI. By the Birkhoff Subdirect Product Theorem  $\mathbf{A} \cong \prod_{i \in I} \mathbf{B}_i$  with  $\mathbf{B}_i$  SDEI for every  $i \in I$ . Since  $\mathbf{A}$  is SDI, there is an  $i \in I$  such that  $\mathbf{A} \cong \mathbf{B}_i$ . Hence  $\mathbf{A}$  is SDEI and thus SDI.  $\square$

As an application of the Birkhoff Subdirect Product Theorem we see that every distributive lattice is isomorphic to a lattice of sets. Recall that a *bounded* lattice has the form  $\langle L, \wedge, \vee, 0, 1 \rangle$  where 0 and 1 are the smallest and largest elements of  $L$ , respectively.

**Theorem 2.73.** *The only SDI distributive lattice is the 2-element lattice  $\mathbf{D}_2$ .*

*Proof.* Let  $\mathbf{L}$  be a bounded distributive lattice, and let  $\text{Co}(\mathbf{L})^+ = \text{Co}(\mathbf{L}) \setminus \{\Delta_L\}$ , the set of congruences of  $\mathbf{L}$  strictly larger than  $\Delta_L$ . If  $\mathbf{L} \cong \mathbf{D}_2$ , then  $\text{Co}(\mathbf{L}) = \{\Delta_L, \nabla_L\}$  and  $\mathbf{L}$  is SDI with monolith  $\nabla_L$ . We assume now that  $|L| > 2$  and show that  $\mathbf{L}$  fails to be SDI by showing

$$(26) \quad \bigcap \text{Co}(\mathbf{L})^+ = \Delta_L.$$

For each  $a \in L$  we define binary relations  $\equiv_a^\vee, \equiv_a^\wedge \subseteq A^2$  as follows. For all  $x, y \in L$ ,

$$x \equiv_a^\vee y \text{ if } x \vee a = y \vee a \quad \text{and} \quad x \equiv_a^\wedge y \text{ if } x \wedge a = y \wedge a.$$

We claim that  $\equiv_a^\vee$  and  $\equiv_a^\wedge$  are both congruences of  $\mathbf{L}$ . The best way to see this is to show they are relation kernels of homomorphisms.

Define  $h_a^\vee: L \rightarrow L$  by setting  $h_a^\vee(x) = x \vee a$ .  $h_a^\vee(x \vee y) = (x \vee y) \vee a = (x \vee a) \vee (y \vee a) = h_a^\vee(x) \vee h_a^\vee(y)$  and  $h_a^\vee(x \wedge y) = (x \wedge y) \vee a = (x \vee a) \wedge (y \vee a) = h_a^\vee(x) \wedge h_a^\vee(y)$ . So  $h_a^\vee \in \text{End}(\mathbf{L}, \mathbf{L})$ , and hence  $\equiv_a^\vee = \text{rker}(h_a^\vee) \in \text{Co}(\mathbf{L})$ .  $\equiv_a^\wedge \in \text{Co}(\mathbf{L})$  by duality.

Note that  $x \leq a$  iff  $x \vee a = a = a \vee a$  iff  $x \equiv_a^\vee a$ . So  $0 / \equiv_a^\vee (= a / \equiv_a^\vee) = L[a]$ , the principal ideal of  $\mathbf{L}$  generated by  $a$ , and dually  $1 / \equiv_a^\wedge = L[a]$ , the principal filter generated by  $a$ . In particular, this gives us that

$$0 < a \text{ implies } \equiv_a^\vee \in \text{Co}(\mathbf{L})^+ \quad \text{and} \quad a < 1 \text{ implies } \equiv_a^\wedge \in \text{Co}(\mathbf{L})^+.$$

Suppose there exist nonzero  $a$  and  $b$  in  $\mathbf{L}$  such that  $a \wedge b = 0$ . Then  $\equiv_a^\vee, \equiv_b^\vee \in \text{Co}(\mathbf{L})^+$ . If  $\langle x, y \rangle \in \equiv_a^\vee \cap \equiv_b^\vee$ , then  $x \vee a = y \vee a$  and  $x \vee b = y \vee b$ . So  $(x \vee a) \wedge (x \vee b) = (y \vee a) \wedge (y \vee b)$ . But by distributivity,  $(x \vee a) \wedge (x \vee b) = (x \wedge x) \vee (a \wedge x) \vee (x \wedge b) \vee (a \wedge b) = x$  since  $a \wedge b = 0$ . Similarly,  $(y \vee a) \wedge (y \vee b) = y$ . So  $\langle x, y \rangle \in \equiv_a^\vee \cap \equiv_b^\vee$  implies  $x = y$ , i.e.,  $\bigcap \text{Co}(\mathbf{L})^+ \subseteq \equiv_a^\vee \cap \equiv_b^\vee = \Delta_L$ .

So in the case  $\mathbf{L}$  contains nonzero  $a, b$  such that  $a \wedge b = 0$  we conclude that  $\mathbf{L}$  is not SDI. So we now assume that  $a \wedge b \neq 0$  for all nonzero  $a$  and  $b$ . Then for all such  $a, b$  we have  $\equiv_{a \wedge b}^\vee \in \text{Co}(\mathbf{L})^+$ . But  $a = a \vee (a \wedge b)$  and  $b = b \vee (a \wedge b)$  by absorption. So, if  $a \neq b$  and both are different from 0 we have  $\langle a, b \rangle \notin \equiv_{a \wedge b}^\vee$  and hence  $\langle a, b \rangle \notin \bigcap \text{Co}(\mathbf{L})^+$ . By duality, the same is true if  $a \neq b$  and both are different from 1. It remains only to consider the case  $a = 0$  and  $b = 1$ . Since  $\mathbf{L} \neq \mathbf{D}_2$ , there is a  $c \in L$  such that  $0 < c < 1$ .  $\langle 0, 1 \rangle \notin \equiv_c^\vee \in \text{Co}(\mathbf{L})^+$ . Thus (26) holds and  $\mathbf{L}$  is not SDI.  $\square$

**Theorem 2.74** (Birkhoff Representation Theorem). *Every distributive lattice is isomorphic to a lattice of sets. More precisely, for every distributive lattice  $\mathbf{L}$  there exists a set  $I$  and an embedding  $h: \mathbf{L} \rightarrow \langle \mathcal{P}(I), \cup, \cap \rangle$ .*

*Proof.* Without loss of generality we can assume  $\mathbf{L}$  is bounded. For if not, then we can adjoin new elements 0 and 1 to  $\mathbf{L}$  and set  $0 < x$  for every  $x \in L \cup \{1\}$  and  $x < 1$  for every  $x \in L \cup \{0\}$ . It is easily checked that the extended po-set obtained this way is a bounded distributive lattice  $\mathbf{L}'$  and that  $\mathbf{L}$  is a sublattice (strictly speaking, a sublattice of the  $\{\wedge, \vee\}$ -reduct of  $\mathbf{L}'$ ). Any embedding of  $\mathbf{L}'$  into a lattice of sets restricts to an embedding of  $\mathbf{L}$  into the same lattice of sets.

By the Birkhoff Subdirect Product Theorem and Thm 2.73 there is a set  $I$  such that  $\mathbf{L} \cong ; \subseteq_{\text{SD}} \mathbf{D}_2^I$ . Let  $g: \mathbf{L} \rightarrow_{\text{SD}} \mathbf{D}_2^I$  be a subdirect embedding. Define  $h: L \rightarrow \mathcal{P}(I)$  by  $h(a) = \{i \in I : g(a)(i) = 1\}$ . Note that  $g(a): I \rightarrow \{0, 1\}$  is the characteristic function of  $h(a)$ . For all  $a, b \in L$  we have

$$\begin{aligned} a \neq b &\iff g(a) \neq g(b) \\ &\iff \exists i \in I (g(a)(i) \neq g(b)(i)) \\ &\iff \exists i \in I (i \in h(a) \setminus h(b) \text{ or } i \in h(b) \setminus h(a)) \\ &\iff h(a) \neq h(b) \end{aligned}$$

So  $h$  is injective. We now check that it is a homomorphism from  $\mathbf{L}$  into  $\langle \mathcal{P}(I), \cup, \cap, \emptyset, I \rangle$ . For every  $i \in I$  we have

$$\begin{aligned} i \in h(a \vee b) &\iff g(a \vee^{\mathbf{L}} b)(i) = 1 \\ &\iff (g(a) \vee^{\mathbf{D}_2^I} g(b))(i) = 1 \\ &\iff (g(a)(i) \vee^{\mathbf{D}_2} g(b)(i)) = 1 \\ &\iff g(a)(i) = 1 \text{ or } g(b)(i) = 1 \\ &\iff i \in h(a) \cup h(b). \end{aligned}$$

So  $h(a \vee b) = h(a) \cup h(b)$ . The equality  $h(a \wedge b) = h(a) \cap h(b)$  is verified in a similar way.  $g(0)$  and  $g(1)$  are respectively the characteristic functions of  $\emptyset$  and  $I$ . So  $h$  is a homomorphism.  $\square$

Exercise: Let  $\mathbf{A}$  be a finitely generated Abelian group. Then  $\mathbf{A}$  is SDI iff  $\mathbf{A} \cong \mathbf{Z}_{p^n}$  for some prime  $p$  and some  $n \in \omega \setminus \{0\}$ . Hint: Use the Fundamental Theorem of Abelian Groups.

Note that by the Fundamental Theorem of Abelian Groups the Birkhoff Subdirect Product Theorem holds in a much stronger form when restricted to finitely generated Abelian groups: every such algebra is isomorphic to a direct product (not just a subdirect product) of subdirectly irreducible algebras.

Every simple algebra is SDI but not conversely, and every SDI algebra is directly indecomposable (DI) but not conversely. The analog of the Birkhoff Subdirect Product Theorem does not hold for direct products. However we do have the following partial result.

**Theorem 2.75.** *If  $\mathbf{A}$  is a finite  $\Sigma$ -algebra, then there exist DI  $\Sigma$ -algebras  $\mathbf{B}_1, \dots, \mathbf{B}_n$  such that  $\mathbf{A} \cong \mathbf{B}_1 \times \dots \times \mathbf{B}_n$ .*

*Proof.* The proof is by induction on  $|A|$ . If  $\mathbf{A}$  itself is DI then we are done. Otherwise  $\mathbf{A} \cong \mathbf{A}' \times \mathbf{A}''$  with  $1 < |A'|, |A''| < |A|$ . By the induction hypothesis  $\mathbf{A}' \cong \mathbf{B}'_1 \times \mathbf{B}'_{n'}$  and  $\mathbf{A}'' \cong \mathbf{B}''_1 \times \mathbf{B}''_{n''}$  with  $\mathbf{B}'_1, \dots, \mathbf{B}'_{n'}, \mathbf{B}''_1, \dots, \mathbf{B}''_{n''}$  DI. Then  $\mathbf{A} \cong \mathbf{B}'_1 \times \dots \times \mathbf{B}'_{n'} \times \mathbf{B}''_1 \times \dots \times \mathbf{B}''_{n''}$ .  $\square$

Every direct factor of a group  $\mathbf{G}$  must also be a group because it is a homomorphic image of  $\mathbf{G}$  and the group identities are clearly preserved under homomorphism. (In the next chapter we will systematically study the preservation of identities.) Every finite group has the *unique direct decomposition property*, i.e., if  $\mathbf{G}$  is finite, and  $\mathbf{G} \cong \mathbf{A}_1 \times \cdots \times \mathbf{A}_n$  and  $\mathbf{G} \cong \mathbf{B}_1 \times \cdots \times \mathbf{B}_m$  with  $\mathbf{A}_1, \dots, \mathbf{A}_n, \mathbf{B}_1, \dots, \mathbf{B}_m$  DI, then  $n = m$  and, after reordering the  $\mathbf{B}_i$  if necessary, we have  $\mathbf{A}_1 \cong \mathbf{B}_1, \dots, \mathbf{A}_n \cong \mathbf{B}_n$ . This is the well know *Krull-Schmidt Theorem* of group theory.

In general however finite algebras do not have the unique direct decomposition property. For example, consider the following two mono-ary algebras  $\mathbf{A}$  and  $\mathbf{B}$  over the same universe  $\{0, 1\}$  such that  $f^{\mathbf{A}}(0) = 1, f^{\mathbf{A}}(1) = 0$  and  $f^{\mathbf{B}}(0) = 0, f^{\mathbf{B}}(1) = 1$ .

Exercise. Show that  $\mathbf{A} \times \mathbf{B} \cong \mathbf{A}^2$  but  $\mathbf{A} \not\cong \mathbf{B}$ .

We now show that there exist infinite algebras that are not isomorphic to any product, finite or infinite, of SD algebras.

Recall that a groupoid  $\mathbf{A} = \langle A, \cdot \rangle$  is a *left-trivial semigroup* if it satisfies the identity  $x \cdot y \approx x$ . Any two left-trivial semigroups  $\mathbf{A}$  and  $\mathbf{B}$  of the same cardinality are isomorphic. Indeed any bijection  $h: A \xrightarrow{\sim} B$  is a homomorphism and hence an isomorphism, for  $h(a \cdot b) = h(a) = h(a) \cdot h(b)$ .

We claim that a left-trivial semigroup  $\mathbf{A}$  is DI iff it is finite and  $|A|$  is a prime. The implication from left to right is obvious. For the implication in the opposite direction, suppose first of all that  $\mathbf{A}$  is finite and composite, say  $|A| = n \cdot m$  with  $1 < n, m < |A|$ . Let  $\mathbf{B}$  and  $\mathbf{C}$  be left-trivial semigroups of cardinality  $n$  and  $m$  respectively. Then  $\mathbf{A} \cong \mathbf{B} \times \mathbf{C}$ . If  $|A|$  is infinite, then  $|A| = |A \times A|$  by set theory. So  $\mathbf{A} \cong \mathbf{A} \times \mathbf{A}$ .

Take  $\mathbf{A}$  to be the (unique) left-trivial semigroup with universe  $\omega$ . Suppose  $\mathbf{A} \cong \prod_{i \in I} \mathbf{B}_i$  with  $1 < |B_i|$  for each  $i \in I$ . The claim is that the index set  $I$  must be finite. Otherwise, we get  $|\omega| = |A| = |\prod_{i \in I} B_i| \geq 2^{|I|} \geq |2^\omega| = |\mathbb{R}|$ , contradicting the fact that the real numbers cannot be enumerated. Thus  $I$  must be finite and hence, for at least one  $i \in I$ , the cardinality of  $B_i$  is infinite and hence cannot be directly indecomposable.

The final thing to do in this section is show how the direct product of a system of multi-sorted algebras is defined.

Let  $\Sigma$  be a multi-sorted signature with sort set  $S$ . Let  $\langle \mathbf{A}_i : i \in I \rangle$  be a system of  $\Sigma$ -algebras, where the universe of each  $\mathbf{A}_i$  is an  $S$ -sorted set  $\langle A_{i,s} : s \in S \rangle$ . The universe  $\langle A_i : i \in I \rangle$  of the direct product  $\prod_{i \in I} \mathbf{A}_i$  is the  $S$ -sorted system of direct products of sets  $\langle \prod_{i \in I} A_{i,s} : s \in S \rangle$ . Let  $\sigma \in \Sigma$  be of type  $s_1, \dots, s_n \rightarrow s$ , and let  $\vec{a}_j \in \prod_{i \in I} A_{i,s_j}$  for  $j = 1, \dots, n$ . Then

$$\sigma^{\prod \mathbf{A}_i}(\vec{a}_1, \dots, \vec{a}_n) = \langle \underbrace{\sigma^{\mathbf{A}_i}(\vec{a}_1(i), \dots, \vec{a}_n(i))}_{s} : i \in I \rangle.$$

### 3. FREE ALGEBRAS, POLYNOMIAL ALGEBRAS, VARIETIES

Almost all the main classes of algebras studied in abstract algebra are either defined by identities or are derived in some natural way from classes that are defined by identities. Groups, rings, fields, vector spaces, modules are examples. In the general theory of algebra a *variety* is any class of algebras of fixed but arbitrary signature that is defined by identities. An algebra is free over a variety if it does not satisfy any identity that is not satisfied by all



**P(K)**: Let  $h: X \rightarrow \mathbf{B} = \prod_{i \in I} \mathbf{C}_i$ , where  $\langle \mathbf{C}_i : i \in I \rangle \in \mathbf{K}^I$ . For every  $i \in I$  there is by assumption a  $(\pi_i \circ h)^*: \mathbf{A} \rightarrow \mathbf{C}_i$  such that  $(\pi_i \circ h)^* \upharpoonright X = \pi_i \circ h$ . By the Categorical Product Property there is an  $h^*: \mathbf{A} \rightarrow \mathbf{B}$  such that, for every  $i \in I$ ,  $\pi_i \circ h^* = (\pi_i \circ h)^*$ . Thus, for every  $x \in X$  and every  $i \in I$  we have  $\pi_i(h^*(x)) = (\pi_i \circ h^*)(x) = (\pi_i \circ h)^*(x) = (\pi_i \circ h)(x) = \pi_i(h(x))$ . Thus, for every  $x \in X$ ,  $h^*(x) = h(x)$ , i.e.,  $h^* \upharpoonright X = h$ .  $\square$

**Theorem 3.3.** *Assume  $\mathbf{A}$  has the UMP over a class  $\mathbf{K}$  of  $\Sigma$ -algebras wrt to  $X$ , and that  $\mathbf{B}$  has the UMP over  $\mathbf{K}$  wrt  $Y$ . If  $\mathbf{A}, \mathbf{B} \in \mathbf{HSP}(\mathbf{K})$  and  $|X| = |Y|$ , then  $\mathbf{A} \cong \mathbf{B}$ . Moreover, for every bijection  $f: X \cong Y$  there is an isomorphism  $f^*: \mathbf{A} \cong \mathbf{B}$  such that  $f^* \upharpoonright X = f$ .*

*Proof.*  $f: X \rightarrow Y$ . So there is an  $f^*: \mathbf{A} \rightarrow \mathbf{B}$  such that  $f^* \upharpoonright X = f$ . Also,  $f^{-1}: Y \rightarrow X$ . So there is a  $(f^{-1})^*: \mathbf{B} \rightarrow \mathbf{A}$  such that  $(f^{-1})^* \upharpoonright Y = f^{-1}$ . Then  $(f^{-1})^* \circ f^*: \mathbf{A} \rightarrow \mathbf{A}$ , and, for every  $x \in X$ ,  $((f^{-1})^* \circ f^*)(x) = (f^{-1})^*(f^*(x)) = (f^{-1})^*(f(x)) = f^{-1}(f(x)) = (f^{-1} \circ f)(x) = x$ . So  $(f^{-1})^* \circ f^* = \Delta_{\mathbf{A}}$  by the uniqueness property. Similarly,  $f^* \circ (f^{-1})^* = \Delta_{\mathbf{B}}$ . So  $(f^{-1})^* = (f^*)^{-1}$  and  $f^*: \mathbf{A} \cong \mathbf{B}$ .  $\square$

Here are two well known examples from abstract algebra of algebras with the UMP.

(1) If  $\mathbf{K}$  is the class of all Abelian groups, then, for every  $n \in \omega$ ,  $\mathbf{Z}^n$  has the UMP over  $\mathbf{K}$  wrt the set of “unit vectors”, i.e.,  $\{\langle 1, 0, \dots, 0 \rangle, \langle 0, 1, 0, \dots, 0 \rangle, \dots, \langle 0, \dots, 0, 1 \rangle\}$ .

(2) If  $\mathbf{K}$  is the class of all vector spaces over a field, then every  $\mathbf{V} \in \mathbf{K}$  has the UMP wrt any basis of  $\mathbf{V}$ .

We now construct for each signature  $\Sigma$  and each set  $X$  such that  $X \cap \Sigma = \emptyset$ , a  $\Sigma$ -algebra that has the UMP over the class of all  $\Sigma$ -algebras wrt  $X$ .

Let  $\text{St}_{\Sigma}(X) = (\Sigma \cup X)^*$ , the set of all finite sequences of elements of  $\Sigma \cup X$ . An element  $\langle a_1, \dots, a_n \rangle$  of  $\text{St}_{\Sigma}(X)$  is called a *string* and written simply as  $a_1 \dots a_n$ . Define for each  $n \in \omega$  and  $\sigma \in \Sigma_n$ ,

$$\begin{aligned} \sigma^{\text{St}_{\Sigma}(X)}(a_{1,1}a_{1,2} \dots a_{1,m_1}, \dots, a_{n,1}a_{n,2} \dots a_{n,m_n}) \\ = a_{1,1}a_{1,2} \dots a_{1,m_1}, \dots, a_{n,1}a_{n,2} \dots a_{n,m_n}. \end{aligned}$$

The  $\Sigma$ -algebra

$$\mathbf{St}_{\Sigma}(X) = \langle \text{St}_{\Sigma}(X), \sigma^{\text{St}_{\Sigma}(X)} \rangle_{\sigma \in \Sigma}$$

is called the *string  $\Sigma$ -algebra in  $X$* .

**Definition 3.4.**  $\text{Te}_{\Sigma}(X) = \text{Sg}^{\text{St}_{\Sigma}(X)}(X)$ . An element of  $\text{Te}_{\Sigma}(X)$  is called a  $\Sigma$ -term in  $X$ ; the corresponding subalgebra of  $\mathbf{St}_{\Sigma}(X)$  is

$$\mathbf{Te}_{\Sigma}(X) = \langle \text{Te}_{\Sigma}(X), \sigma^{\text{Te}_{\Sigma}(X)} \rangle_{\sigma \in \Sigma}$$

is called the  *$\Sigma$ -term algebra in  $X$* .  $X$  is called the set of *variables* of  $\mathbf{Te}_{\Sigma}(X)$ .