2.9. Subdirect products. We give two standard homomorphism constructions involving direct products that are used often in the sequel.

(I) Let \( \langle A_i : i \in I \rangle \) and \( \langle B_i : i \in I \rangle \) be \( I \)-indexed systems of \( \Sigma \)-algebras. Let \( \bar{h} = \langle h_i : i \in I \rangle \in \prod_{i \in I} \text{Hom}(A_i, B_i) \). We denote by \( \prod_i h_i \) or simply by \( \prod \bar{h} \) the homomorphism from \( \prod_{i \in I} A_i \) into \( \prod_{i \in I} B_i \) such that, for every \( \langle a_i : i \in I \rangle \in \prod_{i \in I} A_i \),

\[
(\prod_{i \in I} h_i)(\langle a_i : i \in I \rangle) = \langle h_i(a_i) : i \in I \rangle.
\]

That \( \prod \bar{h} \) is a homomorphism is an immediate consequence of the categorical product property, but it can also be easily verified directly. \( \prod \bar{h} \) is called the product of the system \( \bar{h} \). It is easily checked that \( \prod \bar{h} \) is an epimorphism, a monomorphism, or an isomorphism if, for each \( i \in I \), \( h_i \) has the respective property.

(II) Let \( A \) be a \( \Sigma \)-algebra and let \( \bar{\alpha} = \langle \alpha_i : i \in I \rangle \in \text{Co}(A)^I \). We denote by \( \Delta_{\bar{\alpha}} \) the homomorphism from \( A \) to \( \prod_{i \in I} A/\alpha_i \) such that, for every \( a \in A \),

\[
\Delta_{\bar{\alpha}}(a) = \langle a/\alpha_i : i \in I \rangle.
\]

As in the case of a product of a system of homomorphisms, that \( \Delta_{\bar{\alpha}} \) is a homomorphism can be obtained from the categorical product property or verified directly. \( \Delta_{\bar{\alpha}} \) is called the natural map from \( A \) into \( \prod_{i \in I} A/\alpha_i \).

**Definition 2.61** (Subdirect Product). Let \( \langle B_i : i \in I \rangle \) be a system of \( \Sigma \)-algebras. A subalgebra \( A \) of \( \prod_{i \in I} B_i \) is called a subdirect product of the system \( \langle B_i : i \in I \rangle \), in symbols \( A \subseteq_{\text{sd}} \prod_{i \in I} B_i \), if the projection of \( A \) onto each of the components \( B_i \) is surjective, i.e., for all \( i \in I \), \( \pi_i(A) = B_i \).

If all of the components \( B_i \) of \( \langle B_i : i \in I \rangle \) are the same algebra, say \( B \), then \( A \) is called a subdirect power of \( B \) and we write \( A \subseteq_{\text{sd}} B^I \).

It is helpful to note that \( A \subseteq_{\text{sd}} \prod_{i \in I} B_i \) iff for every \( i \in I \) and every \( b \in B_i \), \( b \) appears as the \( i \)-th component of at least one element of \( A \).

The direct product itself \( \prod_{i \in I} B_i \) is obviously a subdirect product of \( \langle B_i : i \in I \rangle \), and is the largest one. Given any algebra \( B \) and any index set \( I \), let \( D \) be the set of all constant functions from \( I \) into \( B \), i.e., \( D = \{ \langle b, b, \ldots, b \rangle : b \in B \} \). Note that

\[
\sigma^B(\langle b_1, b_1, \ldots, b_n \rangle, \ldots, b_n, b_n, \ldots, b_n) = \langle \sigma^B(b_1, b_2, \ldots, b_n), \sigma^B(b_1, b_2, \ldots, b_n), \ldots, \sigma^B(b_1, b_2, \ldots, b_n) \rangle.
\]

So \( D \) is a nonempty subuniverse of \( B^I \). Clearly for every \( i \in I \) and every \( b \in B \), \( b \) is the \( i \)-component of some (in this case unique) element of \( D \). So \( D \), the subalgebra of \( B^I \) with universe \( D \), is a subdirect power of \( B \). \( D \) is called the \( I \)-th diagonal subdirect power of \( B \) for obvious reasons; it is isomorphic to \( B \). In general it is not the smallest \( I \)-th subdirect power of \( B \). To show this we apply the following lemma, which often proves useful in verifying subdirect products.

**Lemma 2.62.** Let \( \langle B_i : i \in I \rangle \) be a system of \( \Sigma \)-algebras, and let \( X \subseteq \prod_{i \in I} B_i \). Let \( A = \text{Sg}^\prod_B(X) \), the subalgebra of \( \prod_{i \in I} B_i \) generated by \( X \). Then \( A \) is a subdirect product of \( \langle B_i : i \in I \rangle \) iff, for each \( i \in I \), \( B_i = \text{Sg}^B(\pi_i(X)) \).
Proof. By Thm 2.14(iii) \( \pi_i(Sg^\prod B_i(X)) = Sg^B(\sigma_i(X)) \), for each \( i \in I \). \( \square \)

Let \( \langle 1,3 \rangle \in \mathbb{Z}_8 \times \mathbb{Z}_8 \). Since \( \mathbb{Z}_8 \times \mathbb{Z}_8 \) is generated by both 1 and 3, the cyclic subgroup of \( \mathbb{Z}_8 \times \mathbb{Z}_8 \) is a subdirect power of \( \mathbb{Z}_8 \) by the lemma. But it clearly does not include the diagonal subdirect power.

**Definition 2.63 (Subdirect Irreducibility).** A \( \Sigma \)-algebra \( A \) is subdirectly irreducible (SI) if, for every system \( \langle B_i : i \in I \rangle \) of \( \Sigma \)-algebras, \( A \cong \sqsubseteq_{sd} \prod_{i \in I} B_i \) implies \( A \cong B_i \) for some \( i \in I \).

Our goal is to prove the so-called Birkhoff subdirect product theorem that says that every \( \Sigma \)-algebra is a subdirect product of a system of subdirectly irreducible algebras. This is one of the major results in the early development of universal algebra. For this purpose it is useful to consider a characterization of subdirect irreducibility that explicitly involves the monomorphism that gives the subirect embedding. We begin with some preliminary definitions.

A monomorphism \( h: A \rightarrow B \), i.e., an injective homomorphism, is also called an embedding of \( A \) in \( B \). Note that \( h \) is an embedding iff

\[
A \cong h(A) \subseteq B.
\]

A homomorphism \( h: A \rightarrow \prod_{i \in I} B_i \) is said to be subdirect if, for every \( i \in I \), \( \pi_i(h(A)) = B_i \), i.e., the homomorphism \( \pi_i \circ h: A \rightarrow B_i \) is surjective. Note that \( h \) is subdirect iff

\[
A \cong h(A) \subseteq_{sd} B.
\]

Finally, a homomorphism \( h: A \rightarrow \prod_{i \in I} B_i \) is a subdirect embedding if it is both an embedding and subdirect, i.e.,

\[
A \cong h(A) \subseteq_{sd} B.
\]

In this case we write \( h: A \rightarrow_{sd} \prod_{i \in I} B_i \). Clearly, \( A \cong \sqsubseteq_{sd} \prod_{i \in I} B_i \) iff there exists a subdirect embedding \( h: A \rightarrow_{sd} \prod_{i \in I} B_i \).

**Lemma 2.64.** Let \( h: A \rightarrow \prod_{i \in I} B_i \) be an arbitrary homomorphism. Then \( \ker(h) = \bigcap_{i \in I} \ker(\pi_i \circ h) \).

**Proof.**

\[
\langle b, b' \rangle \in \bigcap_{i \in I} \ker(\pi_i \circ h) \iff \forall i \in I (\langle b, b' \rangle \in \ker(\pi_i \circ h))
\]

\[
\iff \forall i \in I (h(b)(i) = h(b')(i))
\]

\[
\iff h(b) = h(b')
\]

\[
\iff \langle b, b' \rangle \in \ker(h).
\]

**Corollary 2.65.** (i) A homomorphism \( h: A \rightarrow \prod_{i \in I} B_i \) is an embedding iff \( \bigcap_{i \in I} \ker(\pi_i \circ h) = 1_A \).

(ii) For every \( \alpha = \langle \alpha_i : i \in I \rangle \in \text{Co}(A) \), the natural map \( \Delta_{\alpha}: A \rightarrow \prod_{i \in I} A/\alpha_i \) is a subdirect embedding iff \( \bigcap_{i \in I} \alpha_i = 1_A \).
Proof. (i). By definition of relation kernel we have that \( h \) is an embedding iff \( \text{rker}(h) = \Delta_A \).

(ii). We first note that the natural map \( \Delta_{\tilde{\alpha}} \) of a system of congruences \( \tilde{\alpha} \) is always subdirect because \( \pi_i \circ \Delta_{\tilde{\alpha}} = \Delta_{\alpha_i} \), and the natural map \( \Delta_{\alpha_i} \) is always surjective. Thus \( \Delta_{\tilde{\alpha}} \) is a subdirect embedding iff it is an embedding, which by the lemma is true iff \( \bigcap_{i \in I} \alpha_i = \Delta_A \) since \( \text{rker}(\pi_i \circ \Delta_{\tilde{\alpha}}) = \alpha_i \) for each \( i \in I \).

In the next theorem we characterize in terms of congruences the systems of algebras in which a given algebra can be subdirectly embeddable. Notice that the characterization differs from the corresponding characterization of those systems for which the given algebra is isomorphic to the direct product only in the absence of the Chinese remainder property.

**Theorem 2.66.** Let \( A \) be a \( \Sigma \)-algebra and let \( \langle B_i : i \in I \rangle \) be a system of \( \Sigma \)-algebras.

Then \( A \cong \mathbb{S}D \prod_{i \in I} B_i \) iff there exists a system \( \tilde{\alpha} = \langle \alpha_i : i \in I \rangle \in \text{Co}(A)^I \) such that

(i) \( \bigcap_{i \in I} \alpha_i = \Delta_A \), and

(ii) for every \( i \in I \), \( A/\alpha_i \cong B_i \).

**Proof.** \( \iff \). Assume (i) and (ii) hold. By (i) and Cor. 2.65(ii), there is a subdirect embedding \( \tilde{\alpha} : A \to \prod_{i \in I} B_i \). Let \( \tilde{h} = \langle h_i : i \in I \rangle \in \prod_{i \in I} \text{Iso}(A/\alpha_i, B_i) \). Then

\( A \to_{\text{sd}} \prod_{i \in I} B_i \cong \prod_{i \in I} \tilde{h} \). Thus \( (\prod_{i \in I} \tilde{h}) \circ \Delta_{\tilde{\alpha}} : A \to_{\text{sd}} \prod_{i \in I} B_i \).

\( \implies \). Suppose \( A \cong \mathbb{S}D \prod_{i \in I} B_i \). Let \( h \) be a subdirect embedding. Let \( \alpha_i = \text{rker}(\pi_i \circ h) \) for each \( i \in I \). Then \( \bigcap_{i \in I} \alpha_i = \Delta_A \) by Cor. 2.65(i). Since \( h \) is subdirect, for each \( i \in I \), \( \pi_i \circ h : A \to B_i \) and hence \( A/\alpha_i \cong B_i \) by the First Isomorphism Theorem.

**Definition 2.67.** A \( \Sigma \)-algebra is subdirectly embedding irreducible (SDEI) if, for every subdirect embedding \( h \to_{\text{sd}} \prod_{i \in I} B_i \), there is an \( i \in I \) such that \( \alpha_i : h : A \cong B_i \).

Subdirect embedding irreducibility trivially implies subdirect irreducibility. For suppose \( A \) is SDEI and \( A \cong \mathbb{S}D \prod_{i \in I} B_i \). Let \( h : A \to_{\text{sd}} \prod_{i \in I} B_i \) be a subdirect embedding. Then \( \alpha_i : h : A \cong B_i \) for some \( i \); in particular \( A \cong B_i \). So \( A \) is SDI.

**Theorem 2.68.** An algebra \( A \) is SDEI iff for every \( \tilde{\alpha} = \langle \alpha_i : i \in I \rangle \in \text{Co}(A)^I \), we have \( \bigcap_{i \in I} \alpha_i = \Delta_A \) only if there is an \( i \in I \) such that \( \alpha_i = \Delta_A \).

**Proof.** \( \implies \). Suppose \( \bigcap_{i \in I} \alpha_i = \Delta_A \). Then by Cor. 2.65(ii), \( \Delta_{\tilde{\alpha}} : A \to_{\text{sd}} \prod_{i \in I} A/\alpha_i \). So there exists an \( i \) such that \( \pi_i \circ \Delta_{\tilde{\alpha}} : A \cong A/\alpha_i \). But \( \pi_i \circ \Delta_{\tilde{\alpha}} = \Delta_A \). So \( \alpha_i = \Delta_A \).

\( \iff \). Let \( h : A \to_{\text{sd}} B \) be a subdirect embedding. For each \( i \in I \) let \( \alpha_i = \text{rker}(\pi_i \circ h) \). We have \( \bigcap_{i \in I} \alpha_i = \Delta_A \) by Cor. 2.65(i) because \( h \) is an embedding. So, for some \( i \), \( \alpha_i = \Delta_A \). Thus \( \pi_i \circ h : A \cong B_i \).

**Corollary 2.69.** A \( \Sigma \)-algebra \( A \) is a SDEI iff the set \( \text{Co}(A) \setminus \{ \Delta_A \} \) of congruences of \( A \) strictly larger than \( \Delta_A \) has a smallest element \( \mu \), i.e., \( \Delta_A \subset \mu \) and, for every \( \alpha \in \text{Co}(A) \) such that \( \Delta_A \subset \alpha \), we have \( \mu \subset \alpha \). A graphical representation of the lattice \( \text{Co}(A) \) of congruences of \( A \) is given in Figure 19. \( \mu \) is called the monolith of \( A \).

**Proof.** \( \iff \). \( \bigcap \{ \alpha \in \text{Co}(A) \setminus \{ \Delta_A \} \} \neq \Delta_A \) since \( A \) is SDEI. This is the monolith \( \mu \) of \( A \).

\( \iff \). Suppose \( \langle \alpha_i : i \in I \rangle \in \text{Co}(A)^I \) and, for each \( i \in I \), \( \alpha_i \neq \Delta_A \). Then, for every \( i \in I \), \( \mu \subset \alpha_i \). Hence \( \Delta_A \subset \mu \subset \bigcap_{i \in I} \alpha_i \).

Using the Correspondence Theorem we can relativize this result to obtain a useful characterization of the quotients of an algebra that are SDEI.
Corollary 2.70. Let $A$ be a $\Sigma$-algebra and let $\alpha \in \text{Co}(A)$. Then the quotient $A/\alpha$ is SDEI iff the set $\{ \beta \in \text{Co}(A) : \alpha \subseteq \beta \} = \text{Co}(A)[\alpha] \setminus \{ \alpha \}$ of all congruences of $A$ strictly including $\alpha$ has a smallest element $\mu_\alpha$, i.e., $\alpha \subseteq \mu_\alpha$ and, for every $\beta \in \text{Co}(A)$ such that $\alpha \subseteq \beta$ we have $\mu_\alpha \subseteq \beta$. A graphical representation of the principal filter of $\text{Co}(A)$ generated by $\alpha$ is given in the left-hand side of Figure 20.

Proof. By the Correspondence Theorem, Thm. 2.26, the map $\beta \mapsto \beta/\alpha$ is an isomorphism between the lattices $\text{Co}(A)[\alpha]$ and $\text{Co}(A/\alpha)$. See Figure 20

If $A/\alpha$ is SDEI, then $A/\alpha$ has a monolith $\mu$. Let $\mu_\alpha$ be the unique congruence in $\text{Co}(A)[\alpha]$ such that $\mu_\alpha/\alpha = \mu$. Then $\mu_\alpha$ is the smallest element of $\text{Co}(A)[\alpha] \setminus \{ \alpha \}$. Conversely, if $\mu_\alpha$ is the smallest element of $\text{Co}(A)[\alpha] \setminus \{ \alpha \}$, then $\mu_\alpha/\alpha$ is the monolith of $A/\alpha$ and hence $A/\alpha$ is SDEI.

Let $L$ be a complete lattice. An element $a \in A$ is strictly meet irreducible (SMI) if, for every $X \subseteq L$, we have that $a = \bigwedge X$ only if $a = x$ for some $x \in X$. Clearly $a$ is SMI iff $a < \bigwedge \{ x \in L : a < x \}$. $A$ is SDEI iff $\Delta_A$ is SMI in the lattice $\text{Co}(A)$; more generally, for every $\alpha \in \text{Co}(A)$, $A/\alpha$ is SDEI iff $\alpha$ is SMI.
**Theorem 2.71** (Birkhoff Sdirect Product Theorem). *Every nontrivial \( \Sigma \)-algebra is isomorphic to a subdirect product of SDEI algebras.*

**Proof.** For all distinct \( a, b \in A \) let \( K(a, b) = \{ \alpha \in \text{Co}(A) : \langle a, b \rangle \notin \alpha \} \). \( K(a, b) \neq \emptyset \) since it contains \( \Delta_A \). Let \( C \subseteq K(a, b) \) be a chain, i.e., a set of congruences in \( K(a, b) \) linearly ordered under inclusion. Then \( \langle a, b \rangle \notin \bigcup C \in \text{Co}(A) \). So \( \bigcup C \in K(a, b) \). By Zorn’s lemma \( K(a, b) \) has a maximal element \( \alpha(a, b) \) (it is not in general unique). The claim is that \( \alpha(a, b) \) is strictly meet irreducible. For each \( \beta \in \text{Co}(A) \) such that \( \alpha(a, b) \subseteq \beta \) we have \( \langle a, b \rangle \in \beta \) by the maximality of \( \alpha(a, b) \). So \( \langle a, b \rangle \in \bigcap\{ \beta \in \text{Co}(A) : \alpha(a, b) \subseteq \beta \} \). Thus \( \alpha(a, b) \subseteq \bigcap\{ \beta \in \text{Co}(A) : \alpha(a, b) \subseteq \beta \} \). So \( \alpha(a, b) \) is SMI and hence \( A/\alpha(a, b) \) is SDEI for all \( \langle a, b \rangle \in A^2 \setminus \Delta_A \) by Cor. 2.70. Moreover, \( \bigcap\{ \alpha(a, b) : \langle a, b \rangle \in A^2 \setminus \Delta_A \} = \Delta_A \). So by Thm. 2.65(ii),

\[
A \cong \bigotimes_{\langle a, b \rangle \in A^2 \setminus \Delta_A} A/\alpha(a, b).
\]

\( \square \)