2.9. **Subdirect products.** We give two standard homomorphism constructions involving direct products that are used often in the sequel.

(I) Let $\langle \boldsymbol{A}_i : i \in I \rangle$ and $\langle \boldsymbol{B}_i : i \in I \rangle$ be I-indexed systems of Σ-algebras. Let $\vec{h} = \langle h_i : i \in I \rangle \in \prod_{i \in I} \mathrm{Hom}(\boldsymbol{A}_i, \boldsymbol{B}_i)$. We denote by $\prod_{i \in I} h_i$ or simply by $\prod \vec{h}$ the homomorphism from $\prod_{i \in I} \boldsymbol{A}_i$ into $\prod_{i \in I} \boldsymbol{B}_i$ such that, for every $\langle a_i : i \in I \rangle \in \prod_{i \in I} A_i$,

$$\Big(\prod_{i \in I} h_i\Big)(\langle a_i : i \in I \rangle) = \langle h_i(a_i) : i \in I \rangle.$$

That $\prod \vec{h}$ is a homomorphism is an immediate consequence of the categorical product property, but it can also be easily verified directly. $\prod \vec{h}$ is called the *product* of the system $\vec{h}$. It is easily checked that $\prod \vec{h}$ is an epimorphism, a monomorphism, or an isomorphism if, for each $i \in I$, h_i has the respective property.

(II) Let $\boldsymbol{A}$ be a Σ-algebra and let $\vec{\alpha} = \langle \alpha_i : i \in I \rangle \in \mathrm{Co}(\boldsymbol{A})^I$. We denote by $\Delta_{\vec{\alpha}}$ the homomorphism from $\boldsymbol{A}$ to $\prod_{i \in I} \boldsymbol{A}/\alpha_i$ such that, for every $a \in A$,

$$\Delta_{\vec{\alpha}}(a) = \langle a/\alpha_i : i \in I \rangle.$$

As in the case of a product of a system of homomorphisms, that $\Delta_{\vec{\alpha}}$ is a homomorphism can be obtained from the categorical product property or verified directly. $\Delta_{\vec{\alpha}}$ is called the *natural map* from $\boldsymbol{A}$ into $\prod_{i \in I} \boldsymbol{A}/\alpha_i$.

Definition 2.61 (Subdirect Product)**.** Let $\langle \boldsymbol{B}_i : i \in I \rangle$ be a system of Σ-algebras. A subalgebra $\boldsymbol{A}$ of $\prod_{i \in I} \boldsymbol{B}_i$ is called a *subdirect product* of the system $\langle \boldsymbol{B}_i : i \in I \rangle$, in symbols $\boldsymbol{A} \subseteq_{\mathrm{SD}} \prod_{i \in I} \boldsymbol{B}_i$, if the projection of $\boldsymbol{A}$ onto each of the components $\boldsymbol{B}_i$ is surjective, i.e., for all $i \in I$, $\pi_i(A) = B_i$.

If all of the components $\boldsymbol{B}_i$ of $\langle B_i : i \in I \rangle$ are the same algebra, say $\boldsymbol{B}$, then $\boldsymbol{A}$ is called a *subdirect power* of $\boldsymbol{B}$ and we write $\boldsymbol{A} \subseteq_{\mathrm{SD}} \boldsymbol{B}^{\boldsymbol{I}}$.

It is helpful to note that $\boldsymbol{A} \subseteq_{\mathrm{SD}} \prod_{i \in I} \boldsymbol{B}_i$ iff for every $i \in I$ and every $b \in B_i$, b appears as the i-th component of at least one element of $\boldsymbol{A}$.

The direct product itself $\prod_{i \in I} \boldsymbol{B}_i$ is obviously a subdirect product of $\langle \boldsymbol{B}_i : i \in I \rangle$, and is the largest one. Given any algebra $\boldsymbol{B}$ and any index set I, let D be the set of all constant functions from I into B, i.e., $D = \{ \langle b, b, \dots, b \rangle : b \in B \}$. Note that

$$\begin{aligned}
&\sigma^{\boldsymbol{B}^{\boldsymbol{I}}}\big(\langle b_1, b_1, \dots, b_n \rangle, \dots, \langle b_n, b_n, \dots, b_n \rangle\big) \\
&\qquad\qquad = \langle \sigma^{\boldsymbol{B}}(b_1, b_2, \dots, b_n), \sigma^{\boldsymbol{B}}(b_1, b_2, \dots, b_n), \dots, \sigma^{\boldsymbol{B}}(b_1, b_2, \dots, b_n)\big).
\end{aligned}$$

So D is a nonempty subuniverse of $\boldsymbol{B}^{\boldsymbol{I}}$. Clearly for every $i \in I$ and every $b \in B$, b is the i-component of some (in this case unique) element of D. So $\boldsymbol{D}$, the subalgebra of $\boldsymbol{B}^{\boldsymbol{I}}$ with universe D, is a subdirect power of $\boldsymbol{B}$. $\boldsymbol{D}$ is called the *I-th diagonal subdirect power* of $\boldsymbol{B}$ for obvious reasons; it is isomorphic to $\boldsymbol{B}$. In general it is not the smallest I-th subdirect power of $\boldsymbol{B}$. To show this we apply the following lemma, which often proves useful in verifying subdirect products.

Lemma 2.62. *Let $\langle \boldsymbol{B}_i : i \in I \rangle$ be a system of Σ-algebras, and let $X \subseteq \prod_{i \in I} B_i$. Let $\boldsymbol{A} = \mathbf{Sg}^{\prod \boldsymbol{B}_i}(X)$, the subalgebra of $\prod_{i \in I} \boldsymbol{B}_i$ generated by X. Then $\boldsymbol{A}$ is a subdirect product of $\langle \boldsymbol{B}_i : i \in I \rangle$ iff, for each $i \in I$, $B_i = \mathrm{Sg}^{\boldsymbol{B}_i}\big(\pi_i(X)\big)$.*

Proof. By Thm 2.14(iii) $\pi_i\big(\mathrm{Sg}^{\prod \boldsymbol{B}_i}(X)\big) = \mathrm{Sg}^{\boldsymbol{B}}\big(\sigma_i(X)\big)$, for each $i \in I$. $\square$

Let $\langle 1, 3\rangle \in \mathbb{Z}_8 \times \mathbb{Z}_8$. Since $\boldsymbol{\mathbb{Z}}_8$ is generated by both 1 and 3, the cyclic subgroup of $\boldsymbol{\mathbb{Z}}_8 \times \boldsymbol{\mathbb{Z}}_8$ is a subdirect power of $\boldsymbol{\mathbb{Z}}_8$ by the lemma. But it clearly does not include the diagonal subdirect power.

Definition 2.63 (Subdirect Irreducibility)**.** A Σ-algebra $\boldsymbol{A}$ is *subdirectly irreducible* (SI) if, for every system $\langle\, \boldsymbol{B}_i : i \in I\,\rangle$ of Σ-algebras, $\boldsymbol{A} \cong\ ;\ \subseteq_{\mathrm{SD}} \prod_{i\in I} \boldsymbol{B}_i$ implies $\boldsymbol{A} \cong \boldsymbol{B}_i$ for some $i \in I$.

Our goal is to prove the so-called Birkhoff subdirect product theorem that says that every Σ-algebra is a subdirect product of a system of subdirectly irreducible algebras. This is one of the major results in the early development of universal algebra. For this purpose it is useful to consider a characterization of subdirect irreducibility that explicitly involves the monomorphism that gives the subirect embedding. We begin with some preliminary definitions.

A monomorphism $h\colon \boldsymbol{A} \rightarrowtail \boldsymbol{B}$, i.e., an injective homomorphism, is also called an *embedding* of $\boldsymbol{A}$ in $\boldsymbol{B}$. Note that h is an embedding iff

$$\boldsymbol{A} \overset{h}{\cong} h(\boldsymbol{A}) \subseteq \boldsymbol{B}.$$

A homomorphism $h\colon \boldsymbol{A} \to \prod_{i\in I} \boldsymbol{B}_i$ is said to be *subdirect* if, for every $i \in I$, $\pi_i\big(h(\boldsymbol{A})\big) = \boldsymbol{B}_i$, i.e., the homomorphism $\pi_i \circ h\colon \boldsymbol{A} \to \boldsymbol{B}_i$ is surjective. Note that h is subdirect iff

$$\boldsymbol{A} \overset{h}{\succcurlyeq} h(\boldsymbol{A}) \subseteq_{\mathrm{SD}} \boldsymbol{B}.$$

Finally, a homomorphism $h\colon \boldsymbol{A} \to \prod_{i\in I} \boldsymbol{B}_i$ is a *subdirect embedding* if it is both an embedding and subdirect, i.e.,

$$\boldsymbol{A} \overset{h}{\cong} h(\boldsymbol{A}) \subseteq_{\mathrm{SD}} \boldsymbol{B}.$$

In this case we write $h\colon \boldsymbol{A} \rightarrowtail_{\mathrm{SD}} \prod_{i\in I} \boldsymbol{B}_i$. Clearly, $\boldsymbol{A} \cong\ ;\ \subseteq_{\mathrm{SD}} \prod_{i\in I} \boldsymbol{B}_i$ iff there exists a subdirect embedding $h\colon \boldsymbol{A} \rightarrowtail_{\mathrm{SD}} \prod_{i\in I} \boldsymbol{B}_i$.

Lemma 2.64. *Let* $h\colon \boldsymbol{A} \to \prod_{i\in I} \boldsymbol{B}_i$ *be an arbitrary homomorphism. Then* $\mathrm{rker}(h) = \bigcap_{i\in I} \mathrm{rker}(\pi_i \circ h)$.

Proof.

$$\begin{aligned}
\langle b, b'\rangle \in \bigcap_{i\in I} \mathrm{rker}(\pi_i \circ h) &\Leftrightarrow \forall i \in I\big(\langle b, b'\rangle \in \mathrm{rker}(\pi_i \circ h)\big)\\
&\Leftrightarrow \forall i \in I\big(h(b)(i) = h(b')(i)\big)\\
&\Leftrightarrow h(b) = h(b')\\
&\Leftrightarrow \langle b, b'\rangle \in \mathrm{rker}(h).
\end{aligned}$$

$\square$

Corollary 2.65. (i) *A homomorphism* $h\colon \boldsymbol{A} \to \prod_{i\in I} \boldsymbol{B}_i$ *is is an embedding iff* $\bigcap_{i\in I} \mathrm{rker}(\pi_i \circ h) = \Delta_A$.

(ii) *For every* $\vec{\alpha} = \langle\, \alpha_i : i \in I\,\rangle \in \mathrm{Co}(\boldsymbol{A})$, *the natural map* $\Delta_{\vec{\alpha}}\colon \boldsymbol{A} \to \prod_{i\in I} \boldsymbol{A}/\alpha_i$ *is a subdirect embedding iff* $\bigcap_{i\in I} \alpha_i = \Delta_A$.

Proof. (i). By definition of relation kernel we have that h is an embedding iff $\mathrm{rker}(h) = \varDelta_A$.

(ii). We first note that the natural map $\varDelta_{\vec{\alpha}}$ of a system of congruences $\vec{\alpha}$ is always subdirect because $\pi_i \circ \varDelta_{\vec{\alpha}} = \varDelta_{\alpha_i}$, and the natural map $\varDelta_{\alpha_i}$ is always surjective. Thus $\varDelta_{\vec{\alpha}}$ is a subdirect embedding iff it is an embedding, which by the lemma is true iff $\bigcap_{i\in I} \alpha_i = \varDelta_A$ since $\mathrm{rker}(\pi_i \circ \varDelta_{\vec{\alpha}}) = \alpha_i$ for each $i \in I$. □

In the next theorem we characterize in terms of congruences the systems of algebras in which a given algebra can be subdirectly embeddable. Notice that the characterization differs from the corresponding characterization of those systems for which the given algebra is isomorphic to the direct product only in the absence of the Chinese remainder property.

Theorem 2.66. *Let $\boldsymbol{A}$ be a $\varSigma$-algebra and let $\langle\, \boldsymbol{B}_i : i \in I \,\rangle$ be a system of $\varSigma$-algebras. Then $\boldsymbol{A} \cong\, ;\, \subseteq_{\mathrm{SD}} \prod_{i\in I} \boldsymbol{B}_i$ iff there exists a system $\vec{\alpha} = \langle\, \alpha_i : i \in I \,\rangle \in \mathrm{Co}(\boldsymbol{A})^I$ such that*

(i) *$\bigcap_{i\in I} \alpha_i = \varDelta_A$, and*
(ii) *for every $i \in I$, $\boldsymbol{A}/\alpha_i \cong \boldsymbol{B}_i$.*

Proof. $\Longleftarrow$. Assume (i) and (ii) hold. By (i) and Cor. 2.65(ii), there is a subdirect embedding $\vec{\alpha}\colon \boldsymbol{A} \rightarrowtail_{\mathrm{SD}} \prod_{i\in I} \boldsymbol{B}_i$. Let $\vec{h} = \langle\, h_i : i \in I \,\rangle \in \prod_{i\in I} \mathrm{Iso}(\boldsymbol{A}/\alpha_i, \boldsymbol{B}_i)$. Then $\boldsymbol{A} \overset{\varDelta_{\vec{\alpha}}}{\rightarrowtail}_{\mathrm{SD}} \prod_{i\in I} \boldsymbol{B}_i \overset{\prod \vec{h}}{\cong} \prod_{i\in I} \boldsymbol{B}_i$. Thus $\bigl(\prod \vec{h}\bigr) \circ \varDelta_{\vec{\alpha}}\colon \boldsymbol{A} \rightarrowtail_{\mathrm{SD}} \prod_{i\in I} \boldsymbol{B}_i$.

$\Longrightarrow$. Suppose $\boldsymbol{A} \cong\, ;\, \subseteq_{\mathrm{SD}} \prod_{i\in I} \boldsymbol{B}_i$. Let h be a subdirect embedding. Let $\alpha_i = \mathrm{rker}(\pi_i \circ h)$ for each $i \in I$. Then $\bigcap_{i\in I} \alpha_i = \varDelta_A$ by Cor. 2.65(i). Since h is subdirect, for each $i \in I$, $\pi_i \circ h\colon \boldsymbol{A} \twoheadrightarrow \boldsymbol{B}_i$ and hence $\boldsymbol{A}/\alpha_i \cong \boldsymbol{B}_i$ by the First Isomorphism Theorem. □

Definition 2.67. A $\varSigma$-algebra is *subdirectly embedding irreducible* (SDEI) if, for every subdirect embedding $h \rightarrowtail_{\mathrm{SD}} \prod_{i\in I} \boldsymbol{B}_i$, there is an $i \in I$ such that $\alpha_i\colon h\colon \boldsymbol{A} \cong \boldsymbol{B}_i$.

Subdirect embedding irreducibility trivially implies subdirect irreducibility. For suppose $\boldsymbol{A}$ is SDEI and $\boldsymbol{A} \cong\, ;\, \subseteq_{\mathrm{SD}} \prod_{i\in I} \boldsymbol{B}_i$. Let $h\colon \boldsymbol{A} \rightarrowtail_{\mathrm{SD}} \prod_{i\in I} \boldsymbol{B}_i$ be a subdirect embedding. Then $\alpha_i \circ h\colon \boldsymbol{A} \cong \boldsymbol{B}_i$ for some i; in particular $\boldsymbol{A} \cong \boldsymbol{B}_i$. So $\boldsymbol{A}$ is SDI.

Theorem 2.68. *An algebra $\boldsymbol{A}$ is* SDEI *iff for every $\vec{\alpha} = \langle\, \alpha_i : i \in I \,\rangle \in \mathrm{Co}(\boldsymbol{A})^I$, we have $\bigcap_{i\in I} \alpha_i = \varDelta_A$ only if there is an $i \in I$ such that $\alpha_i = \varDelta_A$.*

Proof. $\Longrightarrow$. Suppose $\bigcap_{i\in I} \alpha_i = \varDelta_A$. Then by Cor. 2.65(ii), $\varDelta_{\vec{\alpha}}\colon \boldsymbol{A} \rightarrowtail_{\mathrm{SD}} \prod_{i\in I} \boldsymbol{A}/\alpha_i$. So there exists an i such that $\pi_i \circ \varDelta_{\vec{\alpha}}\colon \boldsymbol{A} \cong \boldsymbol{A}/\alpha_i$. But $\pi_i \circ \varDelta_{\vec{\alpha}} = \varDelta_A$. So $\alpha_i = \varDelta_A$.

$\Longleftarrow$. Let $h\colon \boldsymbol{A} \rightarrowtail_{\mathrm{SD}} \boldsymbol{B}Ii$ be a subdirect embedding. For each $i \in I$ let $\alpha_i = \mathrm{rker}(\pi_i\colon h)$. We have $\bigcap_{i\in I} \alpha_i \varDelta_A$ by Cor.2.65(i) because h is an embedding. So, for some i, $\alpha_i = \varDelta_A$. Thus $\pi_i \circ h\colon \boldsymbol{A} \cong \boldsymbol{B}_i$. □

Corollary 2.69. *A $\varSigma$-algebra $\boldsymbol{A}$ is a SDEI iff the set $\mathrm{Co}(\boldsymbol{A}) \setminus \{\varDelta_A\}$ of congruences of $\boldsymbol{A}$ strictly larger than $\varDelta_A$ has a smallest element μ, i.e., $\varDelta_A \subset \mu$ and, for every $\alpha \in \mathrm{Co}(\boldsymbol{A})$ such that $\varDelta_A \subset \alpha$, we have $\mu \subseteq \alpha$. A graphical representation of the lattice $\mathbf{Co}(\boldsymbol{A})$ of congruences of $\boldsymbol{A}$ is given in Figure* 19. *μ is called the* ***monolith****of $\boldsymbol{A}$.*

Proof. $\Longrightarrow$. $\bigcap\bigl\{\, \alpha \in \mathrm{Co}(\boldsymbol{A}) \setminus \{\varDelta_A\} \,\bigr\} \neq \varDelta_A$ since $\boldsymbol{A}$ is SDEI. This is the monolith μ of $\boldsymbol{A}$.

$\Longleftarrow$. Suppose $\langle\, \alpha_i : i \in I \,\rangle \in \mathrm{Co}(\boldsymbol{A})^I$ and, for each $i \in I$, $\alpha_i \neq \varDelta_A$. Then, for every $i \in I$, $\mu \subseteq \alpha_i$. Hence $\varDelta_A \subset \mu \subseteq \bigcap_{i\in I} \alpha_i$. □

Using the Correspondence Theorem we can relativize this result to obtain a useful characterization of the quotients of an algebra that are SDEI.

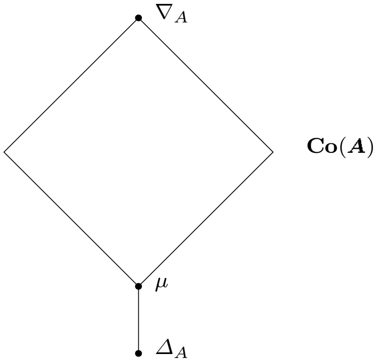

Figure 19

Corollary 2.70. *Let* $\boldsymbol{A}$ *be a* Σ*-algebra and let* $\alpha \in \mathrm{Co}(\boldsymbol{A})$*. Then the quotient* $\boldsymbol{A}/\alpha$ *is* SDEI *iff the set* $\{\, \beta \in \mathrm{Co}(\boldsymbol{A}) : \alpha \subset \beta \,\} = \mathrm{Co}(\boldsymbol{A})[\alpha] \setminus \{\alpha\}$ *of all congruences of* $\boldsymbol{A}$ *strictly including* α *has a smallest element* μ_α*, i.e.,* $\alpha \subset \mu_\alpha$ *and, for every* $\beta \in \mathrm{Co}(\boldsymbol{A})$ *such that* $\alpha \subset \beta$ *we have* $\mu_\alpha \subseteq \beta$*. A graphical representation of the principal filter of* $\mathbf{Co}(\boldsymbol{A})$ *generated by* α *is given in the left-hand side of Figure* 20*.*

Proof. By the Correspondence Theorem, Thm. 2.26, the map $\beta \mapsto \beta/\alpha$ is an isomorphism between the lattices $\mathbf{Co}(\boldsymbol{A})[\alpha]$ and $\mathbf{Co}(\boldsymbol{A}/\alpha)$. See Figure 20

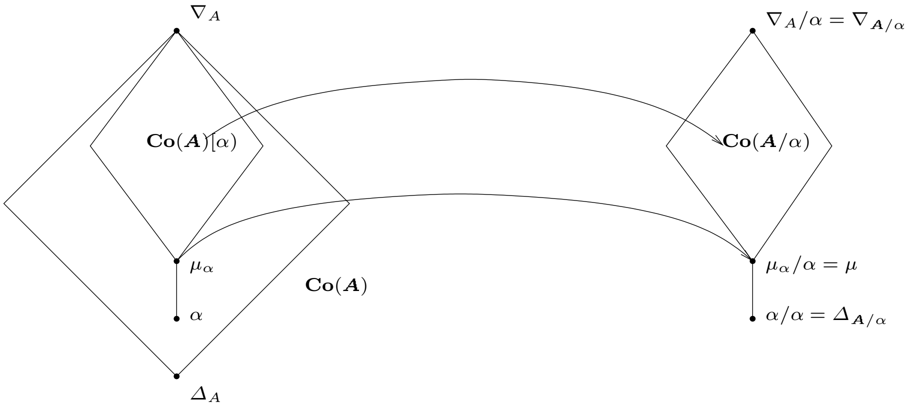

Figure 20

If $\boldsymbol{A}/\alpha$ is SDEI, then $\boldsymbol{A}/\alpha$ has a monolith μ. Let μ_α be the unique congruence in $\mathrm{Co}(\boldsymbol{A})[\alpha)$ such that $\mu_\alpha/\alpha = \mu$. Then μ_α is the smallest element of $\mathrm{Co}(\boldsymbol{A})[\alpha) \setminus \{\alpha\}$. Conversely, if μ_α is the smallest element of $\mathrm{Co}(\boldsymbol{A})[\alpha) \setminus \{\alpha\}$, then μ_α/α is the monolith of $\boldsymbol{A}/\alpha$ and hence $\boldsymbol{A}/\alpha$ is SDEI. □

Let $\boldsymbol{L}$ be a complete lattice. An element $a \in A$ is *strictly meet irreducible* (SMI) if, for every $X \subseteq L$, we have that $a = \bigwedge X$ only if $a = x$ for some $x \in X$ Clearly a is SMI iff $a < \bigwedge\{\, x \in L : a < x \,\}$. $\boldsymbol{A}$ is SDEI iff Δ_A is SMI in the lattice $\mathbf{Co}(\boldsymbol{A})$; more generally, for every $\alpha \in \mathrm{Co}(\boldsymbol{A})$, $\boldsymbol{A}/\alpha$ is SDEI iff α is SMI.

Theorem 2.71 (Birkhoff Sudirect Product Theorem)**.** *Every nontrivial Σ-algebra is isomorphic to a subdirect product of* SDEI *algebras.*

Proof. For all distinct $a, b \in A$ let $K(a,b) = \{\, \alpha \in \mathrm{Co}(\boldsymbol{A}) : \langle a, b\rangle \notin \alpha \,\}$. $K(a,b) \neq \emptyset$ since it contains $\varDelta_A$. Let $C \subseteq K(a,b)$ be a chain, i.e., a set of congruences in $K(a,b)$ linearly ordered under inclusion. Then $\langle a, b\rangle \notin \bigcup C \in \mathrm{Co}(\boldsymbol{A})$. So $\bigcup C \in K(a,b)$. By Zorn's lemma $K(a,b)$ has a maximal element $\alpha(a,b)$ (it is not in general unique). The claim is that $\alpha(a,b)$ is strictly meet irreducible. For each $\beta \in \mathrm{Co}(\boldsymbol{A})$ such that $\alpha(a,b) \subset \beta$ we have $\langle a, b\rangle \in \beta$ by the maximality of $\alpha(a,b)$. So $\langle a, b\rangle \in \bigcap\{\, \beta \in \mathrm{Co}(\boldsymbol{A}) : \alpha(a,b) \subset \beta \,\}$. Thus $\alpha(a,b) \subset \bigcap\{\, \beta \in \mathrm{Co}(\boldsymbol{A}) : \alpha(a,b) \subset \beta \,\}$. So $\alpha(a,b)$ is SMI and hence $\boldsymbol{A}/\alpha(a,b)$ is SDEI for all $\langle a, b\rangle \in A^2 \setminus \varDelta_A$ by Cor. 2.70. Moreover, $\bigcap\{\, \alpha(a,b) : \langle a, b\rangle \in A^2 \setminus \varDelta_A \,\} = \varDelta_A$. So by Thm. 2.65(ii),

$$\boldsymbol{A} \cong ; \subseteq_{\mathrm{SD}} \prod_{\langle a,b\rangle \in A^2 \setminus \varDelta_A} \boldsymbol{A}/\alpha(a,b).$$

□