SOLUTIONS PROBLEM SET # 5

#1. LET $\underline{A} = \langle A, f \rangle$ BE A FINITE CYCLIC MONO-UNARY ALGEBRA GENERATED BY a. LET p BE ITS PERIOD AND ℓ ITS TAIL LENGTH. THUS $|A| = \ell + p - 1$ AND $A = \{ f^n(a) : 0 \le n < \ell + p \}$. FOR ALL k, $\ell \le k < \ell + p$, $f^{k+p}(a) = f^k(a)$.

ASSUME $p > 1$ AND $\ell > 0$. LET k BE THE LEAST NATURAL NUMBER SUCH THAT $\ell \le kp$. DEFINE $h : A \to A$ BY

$$h(f^n(a)) = \begin{cases} f^{pk+n}(a) & \text{IF } n < \ell \text{ AND} \\ f^n(a) & \text{IF } \ell \le n < \ell + p \end{cases}$$

IF $n+1 < \ell$, THEN $h(f(f^n(a))) = h(f^{n+1}(a)) = f^{pk+n+1}(a) = f(f^{pk+n}(a)) = f(h(f^n(a)))$.

IF $n+1 = \ell$,

$h(f(f^n(a))) = h(f^{n+1}(a)) = f^{n+1}(a)$
$= f^{kp+n+1}(a) = f(f^{kp+n}(a)) = f(h(f^n(a)))$

IF $n+1 > \ell$

$h(f(f^n(a))) = h(f^{n+1}(a)) = f^{n+1}(a) =$
$f(f^n(a)) = f(h(f^n(a)))$.

SO h IS AN ENDOMORPHISM OF $\underline{A}$.

LET $\alpha = \text{KER}(h)$. $\alpha \ne \Delta$ SINCE $\ell > 0$

DEFINE $\beta = \{ \langle f^n(a), f^m(a) \rangle : m = n \text{ OR } m, n \ge \ell \}$

$\beta \in Co(\underset{\frown}{A})$ AND $\beta \neq \Delta$ SINCE $p > 1$.
CONSIDER DISTINCT $n, m \leq l + p - 1$. THEN $\langle f^n(a), f^m(a) \rangle \in \alpha$ ONLY IF AT LEAST ONE OF n, m IS $< l$, AND $\langle f^n(a), f^m(a) \rangle \in \beta$ IFF BOTH $n, m \geq l$. SO $\alpha \cap \beta = \Delta$ AND HENCE $\underset{\frown}{A}$ IS NOT SI.

SUPPOSE NOW THAT $p = 1$.
NOTE THAT $f^{l+k}(a) = f^l(a)$ FOR ALL $k \in \omega$. LET $\alpha \in Co(\underset{\frown}{A})$, $\alpha \neq \Delta$.
THEN $\langle f^n(a), f^{n+k}(a) \rangle \in \alpha$ FOR SOME n, k SUCH THAT $n < l$ AND $n + k \leq l$.
THEN $f^n(a) \; \alpha \; f^{n+k}(a) \; \alpha \; f^{n+2k} \; \alpha \ldots$
HENCE $f^n(a) \; \alpha \; f^l(a)$. THEN
$f^{n+1}(a) \; \alpha \; f^{l+1}(a) = f^l(a)$ AND HENCE
$f^m(a) \; \alpha \; f^l(a)$ FOR ALL m, $n \leq m < l$.
SO $\underset{\frown}{A}$ IS SI WITH MONOLITH
$\mu = \Delta_A \cup \{ \langle f^{l-1}(a), f^l(a) \rangle \}$.

NOW ASSUME $l = 0$, IE $\underset{\frown}{A}$ IS A CYCLE OF PERIOD p. FOR EACH DIVISOR k OF p DEFINE $\equiv_k \in A^2$ BY

$$\equiv_k = \{ \langle f^n(a), f^m(a) \rangle : n \leq m < p, \; k \mid n - m \}$$

$$\equiv_k \in Co(\underset{\frown}{A}), \quad \equiv_1 = \nabla_A, \quad \equiv_k \cap \equiv_{k'} = \equiv_{GCD(k,k')}$$

THUS IF $1 < k, k' \mid p$ AND k, k' ARE RELATIVELY

PRIME, THEN $\equiv_h, \equiv_{h'} \neq \Delta$ AND $\equiv_h \cap \equiv_{h'} = \Delta$. SO $\underset{\frown}{A}$ IS NOT SI IF p IS NOT A POWER OF A PRIME. ASSUME $p = q^h$, q - A PRIME AND $h > 0$. THEN $\underset{\frown}{A}$ IS SI AND $\mu = \equiv_{q^{h-1}}$.

FINALLY, WE CONSIDER THE CASE THAT $\underset{\frown}{A}$ IS INFINITE AND CYCLIC, IE, $\underset{\frown}{A}$ IS (UP TO ISOMORPHISM) $\langle \omega, s\rangle$ WITH s THE SUCCESSOR FUNCTION. FOR EACH $h \in \omega$ LET $\equiv_h = \{\langle f^n(a), f^m(a)\rangle : n, m \geq h\}$. $\equiv_h \in Co(\underset{\frown}{A}) - \{\Delta_A\}$ AND $\bigcap_{h\in\omega} \equiv_h = \Delta_A$. SO $\underset{\frown}{A}$ IS NOT SI.

\# 2 (a) LET $\sigma \in \Sigma$, OF RANK n, AND LET $\langle c_1, d_1\rangle, \ldots, \langle c_n, d_n\rangle \in C \times D$. THEN $\sigma^{\underset{\frown}{A}\times\underset{\frown}{B}}(\langle c_1, d_1\rangle, \ldots, \langle c_n, d_n\rangle) = \langle \sigma^{\underset{\frown}{A}}(c_1, \ldots, c_n), \sigma^{\underset{\frown}{B}}(d_1, \ldots, d_n)\rangle = \langle \sigma^{\underset{\frown}{C}}(c_1, \ldots, c_n), \sigma^{\underset{\frown}{D}}(d_1, \ldots, d_n)\rangle \in C \times D$. SO $C \times D \in Sub(\underset{\frown}{A} \times \underset{\frown}{B})$ AND SINCE $C \times D \neq \emptyset$ (BECAUSE $C, D \neq \emptyset$) $\underset{\frown}{C \times D} \leq \underset{\frown}{A} \times \underset{\frown}{B}$.

LET $\underset{\frown}{A}$ BE ANY NONTRIVIAL Σ-ALGEBRA FOR EACH $\sigma \in \Sigma$, OF RANK n, AND ALL

$a_1, \dots, a_n \in A$, $\sigma^{\underset{\frown}{A} \times \underset{\frown}{A}}(\langle a_1, a_1\rangle, \dots, \langle a_n, a_n\rangle)$
$= \langle \sigma^{\underset{\frown}{A}}(a_1, \dots, a_n), \sigma^{\underset{\frown}{A}}(a_1, \dots, a_n)\rangle \in \Delta_A$.
SO $\Delta_A \in \mathrm{SUB}(\underset{\frown}{A})$. SINCE $\Delta_A \neq \emptyset$,
IT IS THE UNIVERSE OF A SUBALGEBRA
$\underset{\frown}{\Delta}_A$ OF $\underset{\frown}{A}$. SINCE $\underset{\frown}{A}$ IS NONTRIVIAL,
$\underset{\frown}{\Delta}_A$ IS NOT OF THE FORM $\underset{\frown}{C} \times \underset{\frown}{D}$.

(b). LET $a_1, \dots, a_n \in C$. SO THERE EXIST
$b_1, \dots, b_n \in D$ SUCH THAT $\langle a_1, b_1\rangle, \dots, \langle a_n, b_n\rangle \in E$
THEN $\langle \sigma^{\underset{\frown}{A}}(a_1, \dots, a_n), \sigma^{\underset{\frown}{B}}(b_1, \dots, b_n)\rangle =$
$\sigma^{\underset{\frown}{A} \times \underset{\frown}{B}}(\langle a_1, b_1\rangle, \dots, \langle a_n, b_n\rangle) \in E$. SO $\sigma^{\underset{\frown}{A}}(a_1, \dots, a_n)$
$\in C$. HENCE $C \in \mathrm{SUB}(\underset{\frown}{A})$. SIMILARLY
$D \in \mathrm{SUB}(\underset{\frown}{B})$. LET $\underset{\frown}{C}, \underset{\frown}{D}$ BE THE SUBALGEBRAS
OF $\underset{\frown}{A}$ AND $\underset{\frown}{B}$ RESPECTIVELY WITH UNIVERSES
C AND D. THEN $\underset{\frown}{E} \leq \underset{\frown}{C} \times \underset{\frown}{D}$.

LET $\langle c, d\rangle \in C \times D$, AND LET $a \in A, b \in B$
SUCH THAT $\langle c, b\rangle, \langle a, d\rangle \in E$. THEN
$\langle c, d\rangle = \langle t^{\underset{\frown}{A}}(c, a), t^{\underset{\frown}{B}}(b, d)\rangle =$
$t^{\underset{\frown}{A} \times \underset{\frown}{B}}(\langle c, b\rangle, \langle a, d\rangle) \in E$. THUS $\underset{\frown}{E} = \underset{\frown}{C} \times \underset{\frown}{D}$.

(c) LET $|A| = n$, $|B| = m$ WITH $\mathrm{GCD}(n, m) = 1$.
LET $k, l \in \mathbb{Z}$ SUCH THAT $kn + lm = 1$.
LET $t(x, y) = x^{lm} y^{kn}$. FOR ALL
$a, b \in A$, $t^{\underset{\frown}{A}}(a, b) = a^{lm} b^{kn} =$
$a^{lm}(b^n)^k = a^{lm} e^k = a^{lm} e = a^{lm} =$
$a^{lm} a^{kn} = a^{kn + lm} = a$. SIMILARLY
$t^{\underset{\frown}{B}}(a, b) = b$ FOR ALL $a, b \in B$.

#3 BY THE CORRESPONDENCE THEOREM THE MAPPING $f : \{\gamma \in Co(\underset{\frown}{A}) : \beta \leq \gamma\} \to Co(\underset{\frown}{A}/\beta)$ SUCH THAT $f(\gamma) = \gamma/\beta$ IS AN ISOMORPHISM BETWEEN THE LATTICES $\underset{\frown}{Co}(\underset{\frown}{A})[\beta)$ AND $\underset{\frown}{Co}(\underset{\frown}{A}/\beta)$.

THUS $\bigcap_{i \in I} \alpha_i/\beta = \bigwedge_{i \in I}^{\underset{\frown}{Co}(A/\beta)} \alpha_i/\beta = \bigwedge_{i \in I}^{Co(A/\beta)} f(\alpha_i) =$

$f\left(\bigwedge_{i \in I}^{Co(\underset{\frown}{A})[\beta)} \alpha_i\right) = f\left(\bigcap_{i \in I} \alpha_i\right) = f(\beta) = \beta/\beta =$

$\Delta_{A/\beta}$.

WE HAD A THEOREM IN CLASS THAT SAID THE FOLLOWING. FOR ANY Σ-ALGEBRA $\underset{\frown}{B}$ AND ANY SYSTEM OF Σ-ALGEBRAS $\langle \underset{\frown}{C}_i : i \in I \rangle$, $\underset{\frown}{B} \cong ; \leq_{SD} \prod_{i \in I} \underset{\frown}{C}_i$ IFF THERE EXISTS $\langle \gamma_i : i \in I \rangle \in Co(\underset{\frown}{B})^I$ SUCH THAT $\bigcap \gamma_i = \Delta_B$ AND $\underset{\frown}{B}/\gamma_i \cong \underset{\frown}{C}_i$ FOR EACH $i \in I$. APPLYING THIS WITH $\underset{\frown}{A}/\beta$ IN PLACE OF $\underset{\frown}{B}$ AND $\langle \alpha_i/\beta : i \in I \rangle$ IN PLACE OF $\langle \gamma_i : i \in I \rangle$ WE GET $\underset{\frown}{A}/\beta \cong ; \leq_{SD} \prod_{i \in I} \underset{\frown}{A}/\beta/\alpha_i/\beta$. BUT $\underset{\frown}{A}/\beta/\alpha_i/\beta \cong \underset{\frown}{A}/\alpha_i$ FOR EACH $i \in I$ BY THE SECOND ISOMORPHISM THEOREM.

AN ALTERNATE, DIRECT PROOF.

BY THE CATEGORICAL PRODUCT PROPERTY THERE IS A HOMOMORPHISM $h : \underset{\frown}{A} \to \prod_{i \in I} \underset{\frown}{A}/\alpha_i$

SUCH THAT $\pi_i \circ h = \Delta_{\alpha_i} : \underset{\sim}{A} \twoheadrightarrow \underset{\sim}{A}/\alpha_i$)
THE NATURAL MAP, FOR EACH $i \in I$.
h IS SUBDIRECT SINCE $\pi_i \circ h$ IS SURJECTIVE FOR EACH $i \in I$. LET $a, a' \in A$. $h(a) = h(a')$ IFF
$\langle a/\alpha_i : i \in I \rangle = \langle a'/\alpha_i : i \in I \rangle$ IFF
$\forall i \in I \, (a/\alpha_i = a'/\alpha_i)$ IFF
$\forall i \in I \, (\langle a, a' \rangle \in \alpha_i)$ IFF
$\langle a, a' \rangle \in \bigcap_{i \in I} \alpha_i \Leftrightarrow \langle a, a' \rangle \in \beta$.
SO $\ker(h) = \beta$. THUS BY THE FIRST ISOMORPHISM THEOREM

$$\underset{\sim}{A}/\beta \cong h(\underset{\sim}{A}) \subseteq_{SD} \prod_{i \in I} \underset{\sim}{A}/\alpha_i .$$

#4. FOR ANY STRING $\alpha = a_1 a_2 \cdots a_m$ WE SAY THAT α HAS PROPERTY (*) IF $\bar{f}(\alpha) = -1$ AND $\bar{f}(\beta) \geq 0$ FOR EVERY PROPER INITIAL SEGMENT β OF α, I.E., $\beta \prec \alpha$.

<u>PROOF OF</u> LEMMA 1. $\Rightarrow$

WE PROVE THAT EVERY $t \in Te_{\Sigma}(X)$ HAS PROPERTY (*) BY STRUCTURAL INDUCTION.
IF $t = x \in X$, THEN $\bar{f}(t) = -1$ AND $\bar{f}(\beta) = 0$ FOR EVERY $\beta \prec t$. (THE EMPTY STRING IS THE ONLY ONE). SUPPOSE $t = \sigma(u_1, \ldots, u_m)$ WITH $\sigma \in \Sigma_m$, $m \geq 1$, AND $u_1, \ldots, u_m \in Te_{\Sigma}(X)$.
$\bar{f}(t) = f(\sigma) + \bar{f}(u_1) + \cdots + \bar{f}(u_m) = (m-1) + m(-1) = -1$
LET $\beta \prec t$. IF $\beta = \sigma$, $\bar{f}(\sigma) = m - 1 \geq 0$

IF $\beta = \sigma \lambda_1 \dots \lambda_i$, $1 \le i < m$, THEN $\bar{f}(\beta) = (m-1) + i(-1) \geq 0$. IF $\beta = \sigma \lambda_1 \dots \lambda_{i-1} \gamma$ WITH $\gamma \prec \lambda_i$, THEN $\bar{f}(\beta) = (m-1) + (i-1)(-1) + \bar{f}(\beta) \geq 0$ SINCE $\bar{f}(\beta) \geq 0$ BY THE INDUCTIVE HYPOTHESIS. IF $t = \sigma \in \Sigma_0$, THEN $\bar{f}(t) = f(\sigma) = -1$ AND $f(\beta) = 0$ FOR $\beta \prec t$.

$\Leftarrow$ LET $\alpha = a_1 \dots a_m$ BE A STRING WITH THE PROPERTY $(*)$. WE PROVE THAT α IS A TERM BY INDUCTION ON THE LENGTH OF α. $\alpha = a_1$. $\bar{f}(\alpha) = f(a_1) = -1$. SO $a_1 \in X \cup \Sigma_0 \subseteq Te_\Sigma(X)$.

ASSUME $m > 1$. BY PROPERTY $(*)$ $f(a_1) \geq 0$. SO $a_1 = \sigma \in \Sigma_m$ WITH $m > 0$. LET a_{i_1} BE THE LEFTMOST SYMBOL SUCH THAT $\bar{f}(a_1 \dots a_{i_1}) = m-2$. IN GENERAL, FOR EACH $k \le m$, LET a_{i_k} BE THE LEFTMOST SYMBOL SUCH THAT $\bar{f}(a_1 \dots a_{i_k}) = m-(k+1)$. THESE MUST ALL EXIST SINCE $\bar{f}$ CAN DECREASE AT MOST 1 AT EACH SYMBOL.

$\bar{f}(a_2 \dots a_{i_1}) = (m-2)-(m-1) = -1$, AND FOR EACH $\beta \prec a_2 \dots a_{i_1}$, $\bar{f}(a_1\beta) > m-2$ SO $\bar{f}(\beta) > (m-2)-(m-1) = -1$. SO $a_2 \dots a_{i_1} \in Te_\Sigma(X)$ BY IND. HYP.

$\bar{f}(a_{i_1+1} \dots a_{i_2}) = (m-3)-(m-2) = -1$, AND FOR EACH $\beta \prec a_{i_1+1} \dots a_{i_2}$, $\bar{f}(a_1 \dots a_{i_1}\beta) > m-3$ SO $\bar{f}(\beta) > (m-3)-(m-2) = -1$. SO $a_{i_1+1} \dots a_{i_2} \in Te_\Sigma(X)$ BY IND. HYP.

IN THIS WAY WE HAVE THAT

$a_{i_{k-1}} \cdots a_{i_k} \in Te_\Sigma(X)$ FOR $k = 1, \ldots, m$.

THUS $\alpha \in Te_\Sigma(X)$. $\boxtimes$ LEMMA 1.

PROOF OF LEMMA 2. $\Leftarrow$ TRIVIAL

$\Rightarrow$ ASSUME $\alpha_1 \cdots \alpha_n = \beta_1 \cdots \beta_m$.

$-n = \bar{f}(\alpha_1) + \cdots + \bar{f}(\alpha_n) = \bar{f}(\beta_1) + \cdots + \bar{f}(\beta_m) = -m$

SO $n = m$. LET $\alpha_1 \cdots \alpha_n = \beta_1 \cdots \beta_m = a_1 \cdots a_k$

WITH $a_1, \ldots, a_k \in X \cup \Sigma$. LET a_i BE THE LEFTMOST SYMBOL SUCH THAT $\bar{f}(a_1 \cdots a_i) = -1$. THEN $\alpha_1 = a_1 \cdots a_i = \beta_1$. THUS $\alpha_2 \cdots \alpha_n = \beta_2 \cdots \beta_n$. HENCE $\alpha_2 = \beta_2, \ldots, \alpha_n = \beta_n$ BY INDUCTION HYPOTHESIS

$\boxtimes$ LEMMA 2

UNIQUE PARSING PROPERTY. TRIVIALLY

$x \neq \sigma(t_1, \ldots, t_n)$ FOR $x \in X$ AND $\sigma \in \Sigma_n$, $t_1, \ldots, t_n \in Te_\Sigma(X)$. SUPPOSE $\sigma(t_1, \ldots, t_n) = \tau(u_1, \ldots, u_m)$. THEN $\sigma = \tau$ AND $t_1 \cdots t_n = u_1 \cdots u_m$. BY LEM 2 $n = m$ AND $t_1 = u_1, \ldots, t_n = u_n$.

#5. (a) THERE IS A BIJECTION BETWEEN $\mathrm{Hom}(\underset{\frown}{\mathrm{Fu}}_\lambda(K), \underset{\frown}{A})$ AND $\{h : X_\lambda^K \to A\}$ BY THE UNIVERSAL MAPPING PROPERTY.

SO $|\mathrm{Hom}(\underset{\frown}{\mathrm{Fu}}_\lambda(K), \underset{\frown}{A})| = |\{h : X_\lambda^K \to A\}|$

$= |A|^{|X_\lambda^K|} = |A|^\lambda$.

SUPPOSE K, λ ARE FINITE AND $\underset{\frown}{A}$ IS FINITE. IF $K \neq \lambda$, THEN $|A|^{\lambda} \neq |A|^{K}$ AND HENCE $|\mathrm{Hom}(\underset{\frown}{\mathrm{Fu}}_{\lambda}(K), \underset{\frown}{A})| \neq |\mathrm{Hom}(\mathrm{Fu}_{K}(K), \underset{\frown}{A})|$. SO $\underset{\frown}{\mathrm{Fu}}_{\lambda}(K) \not\cong \underset{\frown}{\mathrm{Fu}}_{K}(K)$.

(b) LET $\underset{\frown}{F} = \underset{\frown}{\mathrm{Fu}}_{m}(K)$ WITH $m \in \omega$ AND LET $y_0 = x_0^K, \ldots, y_{m-1} = x_{m-1}^K$. LET $Y = \{y_0, \ldots, y_{m-1}\}$. THEN $\underset{\frown}{F}$ HAS THE UMP OVER K WRT TO Y. SINCE K HAS A NONTRIVIAL ALGEBRA, THE $y_0, \ldots, y_{m-1}$ ARE PAIRWISE DISTINCT. LET $Y^- = \{y_0, \ldots, y_{m-2}\}$. AND LET $z_1 = \pi_1^F(y_{m-1})$, $z_2 = \pi_2^F(y_{m-1})$.

<u>CLAIM</u> $\underset{\frown}{F}$ HAS THE UMP OVER K WRT $Y^- \cup \{z_1, z_2\}$.

<u>PROOF</u> $y_{m-1} = \pi_1^F(y_{m-1}) \cdot^F \pi_2^F(y_{m-1})$ SO $Y \subseteq Sg^F(Y^- \cup \{z_1, z_2\})$. THUS $Y^- \cup \{z_1, z_2\}$ GENERATES $\underset{\frown}{F}$. LET $h: Y^- \cup \{z_1, z_2\} \to A$ WHERE $\underset{\frown}{A} \in K$. LET $g: Y \to A$ WHERE $g(y_i) = h(y_i)$ FOR $y_i \in Y^-$ AND $g(y_{m-1}) = h(z_1) \cdot^A h(z_2)$ BY UMP WRT Y, $\exists\, g^*: \underset{\frown}{F} \to \underset{\frown}{A}$

SUCH THAT $g^* \upharpoonright \bar{X} = g$.

$$g^*(z_1) = g^*(\pi_1^{\underset{\frown}{F}}(y_{m-1})) = \pi_1^{\underset{\frown}{A}}(g(y_{m-1})) =$$

$$\pi_1^{\underset{\frown}{A}}(h(z_1) \cdot^{\underset{\frown}{A}} h(z_2)) = h(z_1). \text{ SIMILARLY,}$$

$$g^*(z_2) = \pi_2^{\underset{\frown}{A}}(h(z_1) \cdot^{\underset{\frown}{A}} h(z_2)) = h(z_2).$$

So $g^* \upharpoonright \bar{X}^- \cup \{z_1, z_2\} = h$.

$\boxtimes$ CLAIM

CLAIM $|\bar{X}^- \cup \{z_1, z_2\}| = m+1$

~~PROOF.~~

SUPPOSE $z_1 = y_i$ WITH $i \neq m-1$. THEN $y_i = \pi_1^{\underset{\frown}{F}}(y_{m-1})$. LET $\underset{\frown}{A}$ BE NONTRIVIAL MEMBER OF K AND LET $a, b \in A$, $a \neq b$. LET $h: \bar{X} \to A$ SUCH THAT $h(y_{m-1}) = a$ AND $h(y_i) = b$ IF $\pi_1^{\underset{\frown}{A}}(a) \neq b$ AND $h(y_i) = a$ OTHERWISE. CONSIDER ANY $h^*: \underset{\frown}{F} \to \underset{\frown}{A}$ SUCH THAT $h^*(y_{m-1}) = a$.

THEN $h^*(y_i) = \pi^{\underset{\frown}{A}}(a) \neq h(y_i)$.

THIS CONTRADICTS UMP FOR $\underset{\frown}{F}$ OVER K WRT TO $\bar{X}$. SO $z_1 \notin \bar{X}^-$ AND SIMILARLY $z_2 \notin \bar{X}^-$. SUPPOSE $z_1 = z_2$. LET $h^*: \underset{\frown}{F} \to \underset{\frown}{A}$ SUCH THAT $h^*(y_{m-1}) = a \cdot^{\underset{\frown}{A}} b$.

THEN $h^*(z_1) = \pi_1^{\underset{\frown}{A}}(a \cdot^{\underset{\frown}{A}} b) = a$ AND $h^*(z_2) = \pi_2^{\underset{\frown}{A}}(a \cdot^{\underset{\frown}{A}} b) = b$. THUS $a = b$, A CONTRADICTION $\boxtimes$ CLAIM.

SO $\underset{\frown}{F}$ HAS UMP OVER K WRT A SET OF CARDINALITY $m+1$. SO $\underset{\frown}{\mathrm{Fr}}_m(K) \cong \underset{\frown}{\mathrm{Fr}}_{m+1}(K)$.