

1. [based on Burris-Sanka. II.8.3] Prove that the subdirectly irreducible cyclic (i.e., one-generated) mono-unary algebras are exactly the finite cyclic algebras with empty tail whose cycle is of length a positive power of a prime, or the finite cyclic algebras with nonempty tail whose cycle is of length 1.

[*Hint:* Start by showing that every finite cyclic mono-unary algebra \mathbf{A} with nonempty tail and a cycle of length > 1 fails to be subdirectly irreducible. This is done by finding two congruences that are greater than $\Delta_{\mathbf{A}}$ but intersect in $\Delta_{\mathbf{A}}$. Show that there is an endomorphism of \mathbf{A} that maps all of \mathbf{A} onto its cycle by “wrapping” its tail around its cycle and leaving the cycle itself fixed, and then take the first congruence to be the relation kernel of this map. Show that the equivalence relation that collapses the cycle to one point and leaves the tail alone is a congruence, and take this to be the second congruence. This shows that every subdirectly irreducible finite cyclic algebra is either a cycle of length > 1 with empty tail, or a cycle of length 1 with nonempty tail. You still have to show that algebras of the second kind are SI and that those of the first kind are SI iff the length of the cycle is a power of a prime. Finally, you have to show that the infinite cyclic algebra, i.e., $\langle \omega, s \rangle$ where s is the successor function, is not subdirectly irreducible.

Extra Credit Problem: Describe all SI mono-unary algebras.]

2. Let \mathbf{A} and \mathbf{B} be Σ -algebras.

- (a) Let $\mathbf{C} \subseteq \mathbf{A}$ and $\mathbf{D} \subseteq \mathbf{B}$. Show that $\mathbf{C} \times \mathbf{D} \subseteq \mathbf{A} \times \mathbf{B}$; show by example that in general, not every subalgebra of $\mathbf{A} \times \mathbf{B}$ need be of this form.
- (b) Assume that there is a binary term $t(x_1, x_2)$ such that $t^{\mathbf{A}}(a, a') = a$ for all $a, a' \in A$ and $t^{\mathbf{B}}(b, b') = b'$ for all $b, b' \in B$. Prove that in this case, every subalgebra of $\mathbf{A} \times \mathbf{B}$ is of the form $\mathbf{C} \times \mathbf{D}$ with $\mathbf{C} \subseteq \mathbf{A}$ and $\mathbf{D} \subseteq \mathbf{B}$.
- (c) Let $\mathbf{A} = \langle A, \cdot, {}^{-1}, e \rangle$ and $\mathbf{B} = \langle B, \cdot, {}^{-1}, e \rangle$ be finite groups such that $|A|$ and $|B|$ are relatively prime. Prove that there exists a binary term t satisfying the condition of part (a). Hence every subgroup of $\mathbf{A} \times \mathbf{B}$ is a product of subgroups of \mathbf{A} and \mathbf{B} .

[*Hint for part (b):* Let $\mathbf{E} \subseteq \mathbf{A} \times \mathbf{B}$. Let $C = \{a \in A : \exists b \in B (\langle a, b \rangle \in E)\}$ and let $D = \{b \in B : \exists a \in A (\langle a, b \rangle \in E)\}$. Prove in general (i.e., without the assumption about the existence of the term t) that $C \in \text{Sub}(\mathbf{A})$ and $D \in \text{Sub}(\mathbf{B})$ and that $\mathbf{E} \subseteq \mathbf{C} \times \mathbf{D}$. Now prove that $\mathbf{E} = \mathbf{C} \times \mathbf{D}$; here is where the term t is used.

Prove for every n -ary term $s(x_1, \dots, x_n)$ and for all $\langle a_1, b_1 \rangle, \dots, \langle a_n, b_n \rangle \in \mathbf{A} \times \mathbf{B}$ that

$$s^{\mathbf{A} \times \mathbf{B}}(\langle a_1, b_1 \rangle, \dots, \langle a_n, b_n \rangle) = \langle s^{\mathbf{A}}(a_1, \dots, a_n), s^{\mathbf{B}}(b_1, \dots, b_n) \rangle.$$

Use this together with the assumption on t to show that if $\langle a, b' \rangle, \langle a', b \rangle \in E$, then $\langle a, b \rangle \in E$.]

(over)

3. Let \mathbf{A} be a Σ -algebra, Let $\langle \alpha_i : i \in I \rangle \in \text{Co}(\mathbf{A})^I$, and $\beta = \bigcap_{i \in I} \alpha_i$. Prove that $\mathbf{A}/\beta \cong \prod_{i \in I} \mathbf{A}/\alpha_i$.

[*Hint:* Prove that in the lattice $\mathbf{Co}[\mathbf{A}/\beta]$, $\bigcap_{i \in I} \alpha_i/\beta = \Delta_{\mathbf{A}/\beta}$; for this you can use the Correspondence Theorem.]

4. Let Σ be an arbitrary signature. Prove that the algebra $\mathbf{Te}_\Sigma(X)$ has the unique parsing property wrt X .

[*Hint:* We represent strings over $\Sigma \cup X$ by the Greek letters α, β, γ , possibly with sub- or superscripts. $\alpha\beta$ will denote the concatenation of α and β . α is an *initial segment* of β , in symbols $\alpha \preceq \beta$, if there exists a string γ such that $\beta = \alpha\gamma$. If $\gamma \neq \varepsilon$ (the empty string), then γ is a *proper initial segment* of β .

Define $f: (\Sigma \cup X) \rightarrow \mathbb{Z}$ by setting $f(x) = -1$ for each $x \in X$, and $f(\sigma) = n - 1$ for each $\sigma \in \Sigma_n$. For every string $\alpha = a_1 \dots a_n$ let $\bar{f}(\alpha) = f(a_1) + \dots + f(a_n)$; note $\bar{f}(\varepsilon) = 0$.

Prove the following lemmas.

Lemma 1. *A string α is a Σ -term iff $\bar{f}(\alpha) = -1$ and $\bar{f}(\beta) \geq 0$ for every proper initial segment of α .*

For example

$$\begin{array}{rcccccccc} t : & \wedge & - & \vee & x_1 & 0 & \wedge & x_2 & 1 \\ f : & 1 & 0 & 1 & -1 & -1 & 1 & -1 & -1 \\ \bar{f} : & 1 & 1 & 2 & 1 & 0 & 1 & 0 & -1 \end{array}$$

Lemma 2. *Let $\alpha_1, \dots, \alpha_n$ and β_1, \dots, β_m be two sequences of Σ -terms. $\alpha_1 \dots \alpha_n = \beta_1 \dots \beta_m$ iff $n = m$ and $\alpha_i = \beta_i$ for all $i \leq n$.*

Use this last lemma to prove that $\mathbf{Te}_\Sigma(X)$ has the unique parsing property wrt to X .]

5. (a) Let \mathbf{K} be a variety. Assume that \mathbf{K} contains a nontrivial finite algebra. Prove that, for all cardinals λ and κ , if $\mathbf{Fr}_\lambda(\mathbf{K}) \cong \mathbf{Fr}_\kappa(\mathbf{K})$ iff $\lambda = \kappa$
- (b) Let $\Sigma = \{\cdot, \pi_1, \pi_2\}$ with \cdot a binary operation and π_1 and π_2 unary operations. Let \mathbf{K} be the variety of Σ -algebras defined by the identities $\pi_1(x_1 \cdot x_2) \approx x_2$, $\pi_2(x_1 \cdot x_2) \approx x_1$, and $\pi_1(x) \cdot \pi_2(x) \approx x$. Prove that $\mathbf{Fr}_n(\mathbf{K}) \cong \mathbf{Fr}_m(\mathbf{K})$ for all $n, m \in \omega$ such that $n, m \geq 2$.

[*Hint:* Part (a). Count the number of homomorphisms from $\mathbf{Fr}_\lambda(\mathbf{K})$ and $\mathbf{Fr}_\kappa(\mathbf{K})$ into some nontrivial finite algebra. Part (b): Prove by induction on n that $\mathbf{Fr}_n(\mathbf{K}) \cong \mathbf{Fr}_{n+1}(\mathbf{K})$. For the base step, suppose \mathbf{F} is a free algebra over \mathbf{K} with a single free generator. From this free generator construct two elements such that \mathbf{F} has the UMP over \mathbf{K} wrt these two elements.]