SOLUTIONS PROBLEM SET # 4

1. a) DEFINE $\alpha = \{\langle a,b\rangle : \exists m \in \mathbb{Z}(a-b = m\cdot(c-d))\}$. $a-a = 0(c-d)$ SO α REFLEXIVE. IF $a-b = m(c-d)$, THEN $b-a = (-m)(c-d)$, SO α IS SYMMETRIC. IF $a_0 - a_1 = m(c-d)$ AND $a_1 - a_2 = m(c-d)$ THEN $a_0 - a_2 = (a_0-a_1)+(a_1-a_2) = (m+m)(c-d)$. SO α IS EQUIVALENCE RELATION. SUPPOSE $a_0 - b_0 = m(c-d)$ AND $a_1 - b_1 = m(c-d)$. THEN $(a_0+a_1)-(b_0+b_1) = (a_0-b_0)+(a_1-b_1) = (m+m)(c-d)$; $-a_0-(-b_0) = -(a_0-b_0) = (-m)(c-d)$. SO α HAS SUBSTITUTION PROPERTY AND HENCE $\alpha \in Co(\underset{\sim}{A})$. $c-d = 1(c-d)$ SO $\langle c,d\rangle \in \alpha$ AND HENCE $\Theta_A(c,d) \subseteq \alpha$. SUPPOSE $\langle a,b\rangle \in \alpha$, SAY $a-b = m(c-d)$. THEN $a = b + m(c-d)$. NOTE THAT $c \equiv_{\Theta(c,d)} d$ IMPLIES $c-d \equiv_{\Theta(c,d)} d-d = 0$ AND HENCE $m(c-d) \equiv_{\Theta(c,d)} m0 = 0$ (SINCE $m(c-d) = \underbrace{(c-d)+\cdots+(c-d)}_{m}$ IF $m>0$ AND $m(c-d) = -(-m)(c-d)$ IF $m<0$). THUS $a \equiv_{\Theta(c,d)} b+0 = b$, AND SO $\langle a,b\rangle \in \Theta(c,d)$. THUS $\alpha \subseteq \Theta(c,d)$. SO $\alpha = \Theta_{\underset{\sim}{A}}(c,d)$. LET $\underset{\sim}{B} \leq \underset{\sim}{A}$ AND $b, b' \in B$. $\Theta_{\underset{\sim}{A}}(b,b') \cap B^2$ IS A CONGRUENCE

OF $\underset{\frown}{B}$ THAT INCLUDES $\langle b, b' \rangle$. SO $\Theta_{\underset{\frown}{B}}(b, b') \subseteq \Theta_{\underset{\frown}{A}}(b, b') \cap B^2$. SUPPOSE $\langle a, a' \rangle \in \Theta_{\underset{\frown}{A}}(b, b')$ AND $a, a' \in B$. THEN $a - a' = \hat{m}(b - b')$. THUS, SINCE $a, a' \in B$, $\langle a, a' \rangle \in \Theta_{\underset{\frown}{B}}(b, b')$. SO $\Theta_{\underset{\frown}{A}}(b, b') \cap B^2 = \Theta_{\underset{\frown}{B}}(b, b')$.

b) DEFINE $\alpha = \{\langle a, b \rangle : u \wedge d \wedge a = u \wedge d \wedge b$ AND $u \vee d \vee a = u \vee d \vee b\}$. WE VERIFY α IS A CONGRUENCE RELATION. LET $f_\wedge, f_\vee : A \to A$ BE DEFINED BY $f_\wedge(x) = u \wedge d \wedge x$ AND $f_\vee(x) = u \vee d \vee x$.

<u>CLAIM</u> $f_\wedge, f_\vee \in \mathrm{End}(\underset{\frown}{A})$. $f_\wedge(x \wedge y) = (u \wedge d) \wedge (x \wedge y) = (u \wedge d \wedge x) \wedge (u \wedge d \wedge y) = f_\wedge(x) \wedge f_\wedge(y)$. $f_\wedge(x \vee y) = (u \wedge d) \wedge (x \vee y) = (u \wedge d \wedge x) \vee (u \wedge d \wedge y) = f_\wedge(x) \vee f_\wedge(y)$. SO $f_\wedge \in \mathrm{End}(\underset{\frown}{A})$ AND BY DUALITY $f_\vee \in \mathrm{End}(\underset{\frown}{A})$. $\langle a, b \rangle \in \alpha$ IFF $f_\wedge(a) = f_\wedge(b)$ AND $f_\vee(a) = f_\vee(b)$ IFF $\langle a, b \rangle \in \mathrm{KER}(f_\wedge) \cap \mathrm{KER}(f_\vee)$. SO $\alpha = \mathrm{KER}(f_\wedge) \cap \mathrm{KER}(f_\vee) \in \mathrm{Co}(\underset{\frown}{A})$.

$f_\wedge(u) = u \wedge d = f_\wedge(d)$ AND $f_\vee(u) = u \vee d = f_\vee(d)$. SO $\langle u, d \rangle \in \alpha$ AND HENCE $\Theta_{\underset{\frown}{A}}(u, d) \subseteq \alpha$.

NOTE THAT $u \wedge d \underset{\Theta_{\underset{\frown}{A}}(u,d)}{\equiv} u \wedge u = u = u \vee u$

$\equiv_{\Theta_{\mathbf{A}}(u,d)} u \vee d$. Suppose $\langle a, b\rangle \in \alpha$.

Then $a = (u \wedge d \wedge a) \vee a = (u \wedge d \wedge b) \vee a$
$= ((u \wedge d) \vee a) \wedge (b \vee a) \equiv_{\Theta_{\mathbf{A}}(u,d)} (u \vee d \vee a) \wedge (b \wedge a)$
$= (u \vee d \vee b) \wedge (b \wedge a)$. By symmetry
$b \equiv_{\Theta_{\mathbf{A}}(u,d)} (u \vee d \vee a) \wedge (a \wedge b) = (u \vee d \vee b) \wedge (b \wedge a) \equiv_{\Theta_{\mathbf{A}}(u,d)} a$
and hence $\alpha = \Theta_{\mathbf{A}}(u,d)$.

Let $\mathbf{A} \leq \mathbf{B}$ and $b, b' \in B$.

$\Theta_{\mathbf{B}}(b, b') \subseteq \Theta_{\mathbf{A}}(b, b') \cap B^2$ in general.
Suppose $\langle a, a'\rangle \in \Theta_{\mathbf{A}}(b, b')$ and $a, a' \in B$.
Then $b \wedge^{\mathbf{B}} b' \wedge^{\mathbf{B}} a = b \wedge^{\mathbf{A}} b' \wedge^{\mathbf{A}} a =$
$b \wedge^{\mathbf{A}} b' \wedge^{\mathbf{A}} a' = b \wedge^{\mathbf{B}} b' \wedge^{\mathbf{B}} a'$. Similarly
$b \vee^{\mathbf{B}} b' \vee^{\mathbf{B}} a = b \vee^{\mathbf{B}} b' \vee^{\mathbf{B}} a'$. So $\langle a, a'\rangle \in$
$\Theta_{\mathbf{B}}(b, b')$ and hence $\Theta_{\mathbf{B}}(b, b') =$
$\Theta_{\mathbf{A}}(b, b') \cap B^2$.

#2 (a). Suppose $\nabla_A = \Theta_{\mathbf{A}}(R)$ with $R \subseteq A \times A$ and R finite. Let $\mathcal{C} \subseteq \mathrm{Co}(\mathbf{A}) - \{\nabla_A\}$ such that $\mathcal{C}$ is linearly ordered by $\subseteq$. <u>Claim</u> $\bigcup \mathcal{C} \in \mathrm{Co}(\mathbf{A}) \setminus \{\nabla\}$.
$\bigcup \mathcal{C} \in \mathrm{Co}(\mathbf{A})$ since we know that $\mathrm{Co}(\mathbf{A})$ is an algebraic closed-set system.
If $\bigcup \mathcal{C} = \nabla_A$ then $R \subseteq \bigcup \mathcal{C}$. Thus $R \in \alpha$ for some $\alpha \in \mathcal{C}$ since R finite. So $\alpha = \nabla_A$, a contradiction. So $\bigcup \mathcal{C} \in$

$\mathrm{Co}(\underset{\sim}{A}) - \{\nabla_A\}$. BY ZORN'S LEMMA $\mathrm{Co}(\underset{\sim}{A})$ CONTAINS A MAXIMAL PROPER CONGRUENCE α, IE, α IS COATOM OF THE LATTICE $\underset{\sim}{\mathrm{Co}}(\underset{\sim}{A}) = \langle \mathrm{Co}(\underset{\sim}{A}), \vee, \cap \rangle$. THUS $\underset{\sim}{A}/\alpha$ IS SIMPLE BY CORRESPONDENCE THEOREM AND A HOMOMORPHIC IMAGE OF $\underset{\sim}{A}$.

(b) LET $a \in A$ AND $Y = (X \times \{a\}) \cup (\{\sigma^{\underset{\sim}{A}}(a, a, \ldots, a) : \sigma \in \Sigma\} \times \{a\})$. LET $\alpha = \Theta_{\underset{\sim}{A}}(Y)$. <u>CLAIM</u> $\alpha = \nabla_A$. TO SHOW THIS IT SUFFICES TO SHOW THAT $a/\alpha = A$. $\langle x, a \rangle \in Y \subseteq \alpha$ FOR EVERY $x \in X$. SO $X \subseteq a/\alpha$. SUPPOSE $\sigma \in \Sigma$, $\rho(\sigma) = n$, AND $b_1, \ldots, b_n \in a/\alpha$. THUS $b_i \equiv_\alpha a$ FOR EACH $i \leq n$. SO $\sigma^{\underset{\sim}{A}}(b_1, \ldots, b_n) \equiv_\alpha \sigma^{\underset{\sim}{A}}(a, a, \ldots, a) \equiv_\alpha a$. THUS $\sigma^{\underset{\sim}{A}}(b_1, \ldots, b_n) \in a/\alpha$. HENCE $a/\alpha \in \mathrm{Sub}(\underset{\sim}{A})$ AND SO $a/\alpha = A$ SINCE $X \subseteq a/\alpha$ AND X GENERATES $\underset{\sim}{A}$.

(c) ASSUME $\underset{\sim}{R} = \langle R, +, \cdot, -, 0, 1 \rangle$ IS A COMMUTATIVE RING. LET $\alpha = \Theta_{\underset{\sim}{R}}(\{\langle 0, 1 \rangle\})$. $0 \equiv_\alpha 1$ IMPLIES $0 = 0 \cdot u \equiv_\alpha 1 \cdot u = u$ FOR EVERY $u \in R$. THUS $R = 0/\alpha$ AND HENCE $\alpha = \nabla_R$. NOW APPLY PART (a).

#3 $\Leftarrow$ SUPPOSE EXIST $\langle h_i : i\in I\rangle \in \prod_{i\in I} \mathrm{Hom}(\underset{\sim}{B}, \underset{\sim}{A_i})$ THAT SEPARATES POINTS.
BY CATEGORICAL PRODUCT PROPERTY $\exists!\, h^*: \underset{\sim}{B} \to \prod_{i\in I} \underset{\sim}{A_i}$ SUCH THAT $\pi_i \circ h^* = h_i$ FOR EACH $i\in I$, (I.E., $h^*(b) = \langle h_i(b) : i\in I\rangle$. THUS $h^*(b) = h^*(b')$ IFF $h_i(b) = h_i(b')$ FOR ALL $i\in I$ IFF $b = b'$ SINCE $\langle h_i : i\in I\rangle$ SEPARATES POINTS. SO $h^*: \underset{\sim}{B} \cong h^*(\underset{\sim}{B}) \le \prod \underset{\sim}{A_i}$. THUS $\underset{\sim}{B} \cong ; \le \prod \underset{\sim}{A_i}$.

$\Rightarrow$ SUPPOSE $\underset{\sim}{B} \cong ; \le \prod \underset{\sim}{A_i}$. LET $h^*: \underset{\sim}{B} \hookrightarrow \prod \underset{\sim}{A_i}$. LET $h_i = \pi_i \circ h^*: \underset{\sim}{B} \to \underset{\sim}{A_i}$ FOR ALL $i\in I$. SUPPOSE $b, b' \in B$ WITH $b \neq b'$. THEN $\langle h_i(b) : i\in I\rangle = h^*(b) \neq h^*(b') = \langle h_i(b') : i\in I\rangle$. THUS $h_i(b) \neq h_i(b')$ FOR SOME i. SO $\langle h_i : i\in I\rangle$ SEPARATES POINTS.

#4 $\Rightarrow$ SUPPOSE $\vec{\alpha} = \langle \alpha_1, \dots, \alpha_n\rangle \in Co(\underset{\sim}{A})^n$ IS A FACTOR CONGRUENCE SYSTEM (FCS). THEN $\alpha_1 \cap \cdots \cap \alpha_n = \Delta_A$ BY DEF. OF FCS.
BY THE CRP THE SYSTEM OF CONGRUENCE EQUATIONS $x \equiv_{\alpha_i} a$, $x \equiv_{\alpha_j} b$ FOR ALL $j \le n$, $j \neq i$, HAS A SOLUTION w.

THUS $a \alpha_i c \bigcap_{j \neq i} \alpha_j b$, I.E.,
$\langle a, b \rangle \in \alpha_i ; \hat{\alpha}_i$ FOR EACH $i \leq m$.
SINCE THIS HOLDS FOR ALL $\langle a, b \rangle \in \nabla_A$.
$\alpha_i ; \hat{\alpha}_i = \nabla_A$ FOR EACH $i \leq m$.

$\Leftarrow$ CLAIM. $\alpha_i ; \hat{\alpha}_i = \nabla_A$ FOR EVERY $i \leq m$
IFF $\vec{\alpha}$ HAS THE CRP.
PROOF $\Leftarrow$ THIS WAS PROVED ABOVE
$\Rightarrow$ PROVE BY INDUCTION ON m.
$m = 2$: $\alpha_0 ; \alpha_1 = \nabla_A \Rightarrow \forall a_0, a_1 \in A \, \exists b \in A$
SUCH THAT $a_0 \alpha_0 b \alpha_1 a_1 \Rightarrow \forall a_0, a_1 \in A$
THE PAIR OF CONGRUENCE EQUATIONS
$x \equiv_{\alpha_0} a_0$, $x \equiv_{\alpha_1} a_1$ HAS A SOLUTION.
INDUCTION STEP: SUPPOSE $\vec{\alpha} = \langle \alpha_1, \ldots, \alpha_{m+1} \rangle \in$
$Co(A)^{m+1}$ AND $\alpha_i ; \hat{\alpha}_i = \nabla_A$ FOR EACH
$i \leq m+1$. LET $\tilde{\alpha}_i = \alpha_1 \cap \cdots \cap \alpha_{i-1} \cap \alpha_{i+1} \cap \cdots \cap \alpha_m$
FOR EACH $i \leq m$. $\hat{\alpha}_i \subseteq \tilde{\alpha}_i$ AND HENCE
$\alpha_i ; \hat{\alpha}_i \subseteq \alpha_i ; \tilde{\alpha}_i = \nabla_i$ FOR EACH $i \leq m$.
THUS BY THE IND. HYP. THE SYSTEM OF
CONGRUENCE EQUATIONS $x \equiv_{\alpha_i} a_i$, $i \leq m$,
HAS A SOLUTION c. SINCE $\alpha_m ; \hat{\alpha}_m = \nabla_A$,
THE SYSTEM OF EQUATIONS $x \equiv_{\alpha_{m+1}} a_{m+1}$,
$x \equiv_{\alpha_i} c$ FOR $i \leq m$ HAS A SOLUTION d.
THEN d IS A SOLUTION OF THE $m+1$
CONGRUENCE EQUATIONS $x \equiv_{\alpha_i} a_i$, $i \leq m+1$.
SO $\vec{\alpha} = \langle \alpha_1, \ldots, \alpha_{m+1} \rangle$ HAS THE CRP.
$\boxtimes$ CLAIM
THE DESIRED RESULT FOLLOWS IMMEDIATELY
FROM CLAIM.

#5 a) $\Rightarrow$ ASSUME $\underset{\sim}{A}$ IS DI AND $\underset{\sim}{A} \cong \prod_{i \in I} \underset{\sim}{B}_i$. THEN EXISTS $\langle \alpha_i : i \in I \rangle \in \mathrm{Co}(\underset{\sim}{A})$ SUCH THAT (1) $\bigcap \alpha_i = \Delta_A$
(2) $\langle \alpha_i : i \in I \rangle$ HAS CRP
(3) $\underset{\sim}{A}/\alpha_i \cong \underset{\sim}{B}_i$ FOR EACH $i \in I$.

LET $\hat{\alpha}_i = \bigcap_{j \neq i} \alpha_j$. THEN BY THE CRP WE HAVE

(*) $\qquad \alpha_i ; \hat{\alpha}_i = \nabla_A$ FOR EACH $i \in I$

SO EACH α_i IS A FACTOR CONGRUENCE AND THUS $\alpha_i \in \{\Delta_A, \nabla_A\}$ BY ASSUMPTION. $\alpha_i = \Delta_A$ FOR AT MOST ONE i SINCE, IF $\alpha_i = \alpha_j = \Delta_A$ WITH $i \neq j$, THEN $\hat{\alpha}_i = \Delta_A$, BECAUSE $\hat{\alpha}_i \subseteq \alpha_j$, AND HENCE $\alpha_i ; \hat{\alpha}_i = \Delta_A$ CONTRADICTING (*).

$\Leftarrow$ LET α BE A FACTOR CONGRUENCE OF $\underset{\sim}{A}$ AND LET $\hat{\alpha}$ BE ITS COMPLEMENTARY FACTOR CONGRUENCE. THEN $\alpha \cap \hat{\alpha} = \Delta_A$ AND $\alpha ; \hat{\alpha} = \nabla_A$. THEN $\underset{\sim}{A} \cong \underset{\sim}{A}/\alpha \times \underset{\sim}{A}/\hat{\alpha}$. THUS BY ASSUMPTION $\underset{\sim}{A}/\alpha$ OR $\underset{\sim}{A}/\hat{\alpha}$ IS TRIVIAL, I.E., α OR $\hat{\alpha}$ IS ∇_A. IF $\hat{\alpha} = \nabla_A$ THEN $\alpha = \alpha \cap \nabla_A = \alpha \cap \hat{\alpha} = \Delta_A$. SO $\alpha \in \{\Delta_A, \nabla_{\underset{\sim}{A}}\}$. THIS SHOWS THAT $\underset{\sim}{A}$ IS DI.

b) NOTE THAT, FOR ANY ALGEBRA $\underset{\sim}{A}$, IF $\underset{\sim}{A}$ IS TRIVIAL THEN $\underset{\sim}{A} \times \underset{\sim}{B} \cong \underset{\sim}{B}$ FOR EVERY

ALGEBRA $\underset{\sim}{B}$. ASSUME $\underset{\sim}{A}$ IS FINITE
IF $\underset{\sim}{A} \cong \prod_{i \in I} \underset{\sim}{B}_i$ THEN $\underset{\sim}{B}_i$ CAN BE NONTRIVIAL FOR AT MOST FINITELY MANY i. SO CAN ASSUME $\underset{\sim}{A} \cong \underset{\sim}{B}_1 \times \cdots \times \underset{\sim}{B}_m$.
ASSUME $\underset{\sim}{A}$ IS DI. THEN BY PART (a), FOR SOME $i \le m$, $\underset{\sim}{B}_j$ IS TRIVIAL FOR EVERY $j \neq i$. HENCE $\underset{\sim}{A} \cong \underset{\sim}{B}_i$.

$\Leftarrow$ SUPPOSE $\underset{\sim}{A} \cong \underset{\sim}{B}_1 \times \cdots \times \underset{\sim}{B}_m$ IMPLIES $\underset{\sim}{A} \cong \underset{\sim}{B}_i$ FOR SOME i.
$|A| = |B_1| \cdot \cdots \cdot |B_m|$. THUS

$$|B_1| \cdot \cdots \cdot |B_{i-1}| \cdot |B_{i+1}| \cdot \cdots \cdot |B_m| = \frac{|A|}{|B_i|} = 1.$$

HENCE $\underset{\sim}{B}_j$ IS TRIVIAL FOR EACH $j \neq i$.