Hint on Problem #1

Lets look at the first part. Let Φ be the set of all pairs of integers $\langle a, b\rangle$ such that $a - b = n(c-d)$ for some $n \in \mathbb{Z}$. You must show that $\Phi = \Theta_{\boldsymbol{A}}(a, b)$, i.e., Φ is the smallest congruence α on $\boldsymbol{A}$ such that $a\,\alpha\,b$. First verify by calculation that Φ is a congruence, i.e., an equivalence relation with the substitution property. For example, lets verify it has the substitution property wrt the operation $+$. I.e, if $a_1\,\Phi\,b_1$ and $a_2\,\Phi\,b_2$, then $(a_1{+}a_2)\,\Phi\,(a_1{+}b_2)$. So assume $a_1 - b_1 = n_1(c-d)$ and $a_2 - b_2 = n_2(c-d)$. Then $(a_1 + a_2) - (b_1 + b_2) = (a_1 - b_1) + (a_2 - b_2) = (n_1 + n_2)(c-d)$. Clearly, $c\,\Phi\,d$. This shows that $\Theta_{\boldsymbol{A}}(c, d) \subseteq \Phi$. To show the inclusion in the other direction we must show that if $a\,\Phi\,b$, then $a\,\Theta_{\boldsymbol{A}}(c, d)\,b$. Suppose $a - b = n(c-d)$. Then $a = b + n(c-d)$. Since $c\,\Theta_{\boldsymbol{A}}(c, d)\,d$ and $d\,\Theta_{\boldsymbol{A}}(c, d)\,d$ because $\Theta_{\boldsymbol{A}}(c, d)$ is an equivalence relation, we have by the substitution property that $(c-d)\,\Theta_{\boldsymbol{A}}(c, d)\,(d-d) = 0$. So again by the substitution property, $a = (b + n(c-d))\,\Theta_{\boldsymbol{A}}(c, d)\,(b + n(0)) = b$. So $\Phi \subseteq \Theta_{\boldsymbol{A}}(c, d)$.

Use the same method in part (b). Define a binary relation Φ on the lattice by the given condition. Prove it is a congruence that relates c and d and then prove it is included in $\Theta_{\boldsymbol{L}}(c, d)$