Math 588, Fall 2001 **Problem Set 4** **Due Nov 13, 2001**

1. [based on Burris-Sanka. 1.5.10,1.5.11.] A Σ-algebra $\boldsymbol{A}$ has the *principal congruence extension property* (PCEP) if, for every $\boldsymbol{B} \subseteq \boldsymbol{A}$ and all $b, b' \in B$, $\Theta_{\boldsymbol{B}}(b, b') = \Theta_{\boldsymbol{A}}(b, b') \cap B^2$. A class K of Σ-algebras has the PCEP if every algebra in the class has the PCEP.

   (a) If $\boldsymbol{A}$ is an Abelian group and $a, b, c, d \in A$, show that $\langle a, b\rangle \in \Theta_{\boldsymbol{A}}(c, d)$ iff $a-b = n(c-d)$ for some $n \in \mathbb{Z}$ (i.e., $a - b$ is in the cyclic subgroup generated by $c - d$). Use this to show that the the class of Abelian groups has the PCEP.

   (b) If $\boldsymbol{L}$ is a distributive lattice and $a, b, c, d \in L$, show that $\langle a, b\rangle \in \Theta(c, d)$ iff $c \wedge d \wedge a = c \wedge d \wedge b$ and $c \vee d \vee a = c \vee d \vee b$. Use this to show that the class of distributive lattices has the PCEP.

   Note: A Σ-algebra $\boldsymbol{A}$ has the *congruence extension property* (PCEP) if, for every $\boldsymbol{B} \subseteq \boldsymbol{A}$ and every $\beta \in \mathrm{Co}(\boldsymbol{B})$ there is a $\alpha \in \mathrm{Co}(\boldsymbol{A})$ such that $\beta = \alpha \cap B^2$. A class K of Σ-algebras has the CEP if every algebra in the class has the CEP.

   It is easy to see that the CEP implies the PCEP. We shall see later that the converse holds.

2. Let $\boldsymbol{A}$ be a Σ-algebra.

   (a) Let $\boldsymbol{A}$ be a nontrivial Σ-algebra. Prove that if ∇_A is finitely generated as a congruence, then $\boldsymbol{A}$ has at least one simple homomorphic image.

   (b) Prove that if Σ is finite (i.e., has only a finite number of operation symbols) and A is finitely generated as a subuniverse of itself, then ∇_A is finitely generated as a congruence. Hence any finitely generated nontrivial algebra over a finite language type has a simple homomorphic image.

   (c) Show that any nontrivial ring with unit $\langle R, +, \cdot, -, 0, 1\rangle$ has a field as a homomorphic image.

   [*Hint:* Use Zorn's lemma for the first part.

   For any nonempty subset X of A choose an fixed but arbitrary $a \in X$ and let

   $$Y = \big(X \times \{a\}\big) \cup \big(\{\, \sigma^{\boldsymbol{A}}(a, a, \cdots, a) : \sigma \in \Sigma\,\} \times \{a\}.$$

   For the second part show that $A = \mathrm{Sub}^{\boldsymbol{A}}(X)$ implies $\Theta_{\boldsymbol{A}}(Y) = \nabla_A$.

   For the third part, use part (a). You can use the fact that any simple ring is a field.]

3. Let $\langle\, \boldsymbol{A}_i : i \in I \,\rangle$ be system of Σ-algebras. Let $\boldsymbol{B}$ be a Σ-algebra and $\langle\, h_i : i \in I \,\rangle \in \prod_{i\in I} \mathrm{Hom}(\boldsymbol{B}, \boldsymbol{A}_i)$. $\langle\, h_i : i \in i \,\rangle$ is said to *separate points* if, for all distinct $b, b' \in B$ there exists an $i \in I$ such that $h_i(b) \neq h_i(b')$.

   Prove that $\boldsymbol{B} \cong \boldsymbol{;} \subseteq \prod_{i\in I} \boldsymbol{A}_i$ iff there exists a $\langle\, h_i : i \in I \,\rangle \in \prod_{i\in I} \mathrm{Hom}(\boldsymbol{B}, \boldsymbol{A}_i)$ that separates points.

   [*Hint:* One of the two implications is an immediate corollary of Theorem 7.15 of Burris and Sankappanavar.]

4. Let $\vec{\alpha} = \langle \alpha_1, \dots, \alpha_n \rangle$ be a finite system of congruences on $\boldsymbol{A}$. For each $i \leq n$ let

$$\hat{\alpha}_i = \alpha_1 \cap \cdots \cap \alpha_{i-1} \cap \alpha_{i+1} \cap \cdots \cap \alpha_n.$$

   Prove that $\vec{\alpha}$ is a factor congruence system for $\boldsymbol{A}$ iff

   (a) $\alpha_1 \cap \cdots \cap \alpha_n = \Delta_A$, and
   (b) $\alpha_i \boldsymbol{;} \hat{\alpha}_i = \nabla_A$, for each $i \leq n$.

   [*Hint:* Prove by induction on n that (b) holds iff $\vec{\alpha}$ has the Chinese Remainder Property.]

5. Let $\boldsymbol{A}$ be a Σ-algebra.

   (a) Prove that $\boldsymbol{A}$ is directly irreducible iff, for every system $\langle\, \boldsymbol{B}_i : i \in I \,\rangle$ of Σ-algebras, if $A \cong \prod_{i\in I} \boldsymbol{B}_i$, then there is an $i \in I$ such that $\boldsymbol{B}_j$ is trivial for all $j \in I \setminus \{i\}$.
   (b) Prove that if $\boldsymbol{A}$ is finite, then $\boldsymbol{A}$ is directly irreducible iff, for every system $\langle\, \boldsymbol{B}_i : i \in I \,\rangle$ of Σ-algebras, if $A \cong \prod_{i\in I} \boldsymbol{B}_i$, then there is an $i \in I$ such that $\boldsymbol{A} \cong \boldsymbol{B}_i$.

   [*Hint:* Recall that we define an algebra to be directly irreducible if its only factor congruences are Δ and ∇. Burris and Sankappanavar take the condition in (a) in the special case $I = \{1, 2\}$ to be the definition of direct irreducibility. Thus (a) in this case is their Corollary 7.7. They claim it is a corollary of their Theorems 7.3 and 7.5, but some work is still needed to get it.]