SOLUTIONS TO PROBLEM SET #3

#1 WE PROVE THAT $Sg^{\underline{A}}(\underline{X}) = \bigcup_{n\in\omega} E_n(\underline{X})$.

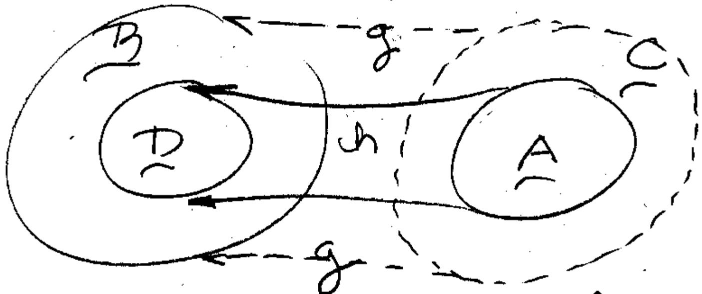
$\supseteq$: WE PROVE THAT $E_n(\underline{X}) \subseteq Sg^{\underline{A}}(\underline{X})$ BY INDUCTION ON n:

$n=0$: $E_0(\underline{X}) = \underline{X} \subseteq Sg^{\underline{A}}(\underline{X})$.

SUPPOSE $E_n(\underline{X}) \subseteq Sg^{\underline{A}}(\underline{X})$ (IND. HYP.). AND LET $a \in E_{n+1}(\underline{X})$. IF $a \in E_n(\underline{X})$, THEN $a \in Sg^{\underline{A}}(\underline{X})$ BY IND. HYP. SO CAN ASSUME THAT $a = \sigma^{\underline{A}}(b_1, \dots, b_m)$ WITH $b_1, \dots, b_m \in E_n(\underline{X})$. SO $b_1, \dots, b_m \in Sg^{\underline{A}}(\underline{X})$ BY IND. HYP. AND HENCE $a \in Sg^{\underline{A}}(\underline{X})$ SINCE $Sg^{\underline{A}}(\underline{X})$ IS CLOSED UNDER $\sigma^{\underline{A}}$.

$\subseteq$: $\underline{X} = E_0(\underline{X}) \subseteq \bigcup E_n(\underline{X})$. FOR CONSIDER ANY $\sigma \in \Sigma_m$ AND $b_1, \dots, b_m \in \bigcup E_n(\underline{X})$. THERE EXISTS A $n \in \omega$ SUCH THAT $b_1, \dots, b_m \in E_n(\underline{X})$. THUS $\sigma^{\underline{A}}(b_1, \dots, b_m) \in E_{n+1}(\underline{X})$. SO $\bigcup E_n(\underline{X}) \in \mathrm{SUB}(\underline{A})$. SINCE $\underline{X} \subseteq \bigcup E_n(\underline{X})$ WE GET $Sg^{\underline{A}}(\underline{X}) \subseteq \bigcup E_n(\underline{X})$.

#2. WE SHOW FIRST THE "VICE VERSA" PART. ASSUME $\underset{\sim}{D} \subseteq \underset{\sim}{B}$ AND $h : \underset{\sim}{A} \twoheadrightarrow \underset{\sim}{D}$.



WE CONSTRUCT $\underset{\sim}{C} \supseteq \underset{\sim}{A}$ AND $g : \underset{\sim}{C} \twoheadrightarrow \underset{\sim}{B}$ SUCH THAT $g \restriction A = h$. WLOG ASSUME $(B \setminus D) \cap A = \emptyset$. TAKE $C = A \cup (B \setminus D)$. FOR ALL $w, w' \in C$ DEFINE $w \cdot^{\underset{\sim}{C}} w' \in C$ AS FOLLOWS. IF $w, w' \in B \setminus D$ AND $w \cdot^{\underset{\sim}{B}} w' \in B \setminus D$, DEFINE $w \cdot^{\underset{\sim}{C}} w' = w \cdot^{\underset{\sim}{B}} w'$; IF $w \cdot^{\underset{\sim}{B}} w' \in D$ TAKE $w \cdot^{\underset{\sim}{C}} w' = a$ WHERE a IS ANY ELEMENT OF A SUCH THAT $h(a) = w \cdot^{\underset{\sim}{B}} w'$ (a EXISTS SINCE $h(A) = D$). IF $w, w' \in A$, TAKE $w \cdot^{\underset{\sim}{C}} w' = w \cdot^{\underset{\sim}{A}} w'$.

SUPPOSE NOW THAT $w \in B \setminus D$ AND $w' \in A$ AND CONSIDER THE ELEMENT $w \cdot^{\underset{\sim}{B}} h(w')$. IF IT IS IN $B \setminus D$, TAKE $w \cdot^{\underset{\sim}{C}} w' = w \cdot^{\underset{\sim}{B}} h(w')$; OTHERWISE TAKE $w \cdot^{\underset{\sim}{C}} w' = a$ WHERE a IS ANY ELEMENT OF $\underset{\sim}{A}$ SUCH

THAT $h(a) = w \cdot^{B} h(w')$.
$w \cdot^{C} w'$ IS SIMILARLY DEFINED IF $w \in A$ AND $w' \in B - D$.

THIS DEFINES THE ALGEBRA $\underset{\sim}{C}$. CLEARLY $\underset{\sim}{A} \subseteq \underset{\sim}{C}$. DEFINE $g : C \to B$ BY

$$g(w) = \begin{cases} w & \text{IF } w \in B \setminus D \\ h(w) & \text{IF } w \in A. \end{cases}$$

g IS SURJECTIVE BECAUSE h IS.

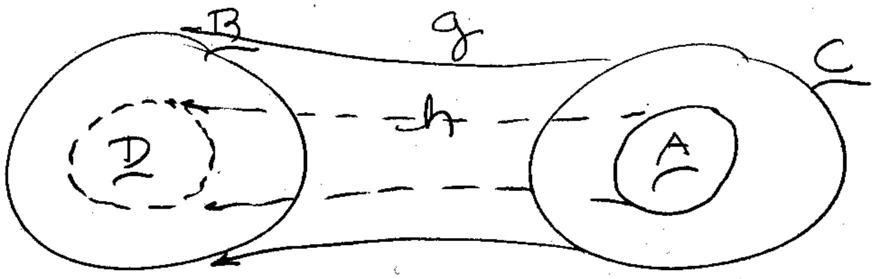
SUPPOSE $w, w' \in B - D$. IF $w \cdot^{B} w' \in B - D$, THEN $g(w \cdot^{C} w') = g(w \cdot^{B} w') = w \cdot^{B} w' = g(w) \cdot^{B} g(w')$.
IF $w \cdot^{B} w' \in D$, THEN $g(w \cdot^{C} w') = g(a) = h(a) = w \cdot^{B} w' = g(w) \cdot^{B} g(w')$.

SUPPOSE $w, w' \in A$. THEN $g(w \cdot^{C} w') = g(w \cdot^{A} w') = h(w \cdot^{A} w') = h(w) \cdot^{B} h(w')$
$= g(w) \cdot^{B} g(w') = g(w) \cdot^{B} g(w')$.

SUPPOSE $w \in B - D$, $w' \in A$. IF $w \cdot^{B} h(w') \in B \setminus D$, THEN $g(w \cdot^{C} w') = g(w \cdot^{B} h(w')) = w \cdot^{B} h(w') = g(w) \cdot^{B} g(w')$. IF $w \cdot^{B} h(w') \in D$, $g(w \cdot^{C} w') = g(a) = h(a) = w \cdot^{B} h(w') = g(w) \cdot^{B} g(w')$. IF $w \in A$ AND

$u' \in B \setminus D$, WE GET $g(u \cdot^{\underset{\sim}{C}} u') = g(u) \cdot^{\underset{\sim}{B}} g(u')$ IN A SIMILAR MANNER. THUS $g : \underset{\sim}{C} \twoheadrightarrow \underset{\sim}{B}$. OBVIOUSLY $g \restriction A = h$.

NOW FOR THE EASIER IMPLICATION



ASSUME $\underset{\sim}{A} \subseteq \underset{\sim}{C}$ AND $g : \underset{\sim}{C} \twoheadrightarrow \underset{\sim}{B}$. TAKE $D = h(\underset{\sim}{A})$ AND $h = g \restriction A$. THEN $h : \underset{\sim}{A} \twoheadrightarrow \underset{\sim}{D}$ AND $\underset{\sim}{D} \subseteq \underset{\sim}{B}$.

#3 Assume $\underset{\frown}{A}$ satisfies entropic law.

Let $f, g \in \mathrm{End}(\underset{\frown}{A})$.

$$(f \cdot g)(a \cdot b) = f(a \cdot b) \cdot g(a \cdot b) =$$
$$(f(a) \cdot f(b)) \cdot (g(a) \cdot g(b)) =$$
$$(f(a) \cdot g(a)) \cdot (f(b) \cdot g(b)) =$$
$$(f \cdot g)(a) \cdot (f \cdot g)(b).$$

Assume $\underset{\frown}{A}$ has an identity element e. Then for all $a, b, c \in A$ we have
$$(a \cdot b) \cdot c = (a \cdot b) \cdot (e \cdot c) = (a \cdot e) \cdot (b \cdot c)$$
$$= a \cdot (b \cdot c)$$ and $$a \cdot b = (e \cdot a) \cdot (b \cdot e) = (e \cdot b) \cdot (a \cdot e) = b \cdot a.$$ So in the presence of an identity the entropic law implies the commutative and associative laws. Clearly the converse always holds.

#4. Suppose $\underset{\frown}{A} = \langle A, \cdot, e \rangle$ is cyclic, say $A = Sg^{\underset{\frown}{A}}(\{a\})$. Define a^n for $n \in \omega$ by recursion. $a^0 = e$, $a^{n+1} = (a^n) \cdot a$.

$$a^n \cdot a^m = a^{n+m} \quad \forall\, n, m \in \omega$$

(Proof by induction on m: $a^n \cdot a^0 = a^n = a^{n+0}$
$$a^n \cdot a^{m+1} = a^n \cdot (a^m \cdot a) = (a^n \cdot a^m) \cdot a \underset{\text{IND HYP}}{=} (a^{n+m}) \cdot a = a^{n+(m+1)}$$
)

LET $B = \{a^n : n \in \omega\}$. $e = a^0 \in B$

$a^n, a^m \in B \Rightarrow a^n \cdot a^m = a^{n+m} \in B$.

SO $B \in \mathrm{SUB}(\underset{\sim}{A})$. THUS $A \subseteq B$ AND HENCE $A = \{a^n : n \in \omega\}$.

$a^n \cdot a^m = a^{n+m} = a^{m+n} = a^m \cdot a^n$.

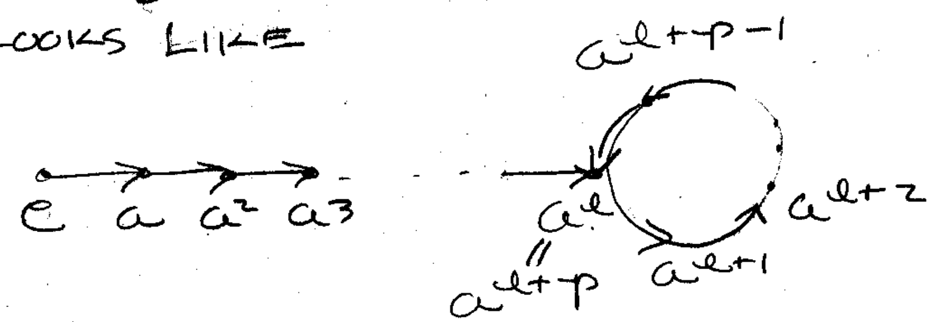
SO $\underset{\sim}{A}$ IS COMMUTATIVE.

THE MAPPING $n \mapsto a^n$ IS AN EPIMORPHISM FROM $\langle \omega, +, 0 \rangle$ ONTO $\underset{\sim}{A}$
($n+m \mapsto a^{n+m} = a^n \cdot a^m$ AND
$0 \mapsto a^0 = e$)

SO $H(\langle \omega, +, 0 \rangle)$ IS THE CLASS OF ALL CYCLIC MONOIDS.

LET $\underset{\sim}{A}$ BE A CYCLIC MONOID GENERATED BY a. SUPPOSE THERE EXIST $n, m \in \omega$, $n \neq m$ SUCH THAT $a^n = a^m$. LET l BE THE LEAST ELEMENT OF ω SUCH THAT $a^{l+k} = a^l$ FOR SOME $k > 0$. THEN LET p BE THE LEAST ELEMENT OF $\omega - \{0\}$ SUCH THAT $a^{l+p} = a^l$.

$\underset{\sim}{A}$ LOOKS LIKE



LET $\equiv_{\ell,p} \subseteq \omega^2$ BE DEFINED BY

$n \equiv_{\ell,p} m$ IFF $n = m$ OR $n \neq m$ AND $n, m \geq \ell$ AND $p \mid (n-m)$.

THEN $\equiv_{\ell,p} = \text{KER}(h)$ WHERE h IS THE UNIQUE EPIMORPHISM FROM $\langle \omega, +, 0 \rangle$ ONTO A. BY THE ABOVE RESULT AND THE FIRST ISOMORPHISM THEOREM WE HAVE $\text{Co}(\langle \omega, +, 0 \rangle) = \{ \equiv_{\ell,p} : \langle \ell, p \rangle \in \omega \times \omega \setminus \{0\} \}$

#5 (a) SUPPOSE $b \vee a = 1$ AND $b \wedge a = 0$. THEN $-a = -a \wedge 1 = -a \wedge (b \vee a) = (-a \wedge b) \vee (-a \wedge a) = (-a \wedge b) \vee 0 = -a \wedge b$ SO $-a \leq b$. SAME CHAIN OF EQUALITIES WITH $-a$ AND b INTERCHANGED GIVES $b \leq -a$. SO $b = -a$ AND HENCE THE COMPLEMENT IS UNIQUE. DOUBLE NEGATION: $--a \vee -a = 1$ AND $--a \wedge -a = 0$ ALSO $a \vee -a = 1$ AND $a \wedge -a = 0$. SO BOTH $--a$ AND a ARE COMPLEMENTS OF $-a$ AND HENCE EQUAL. DEMORGAN LAWS.

$$(-a \wedge -b) \vee (a \vee b) = (-a \vee (a \vee b)) \wedge (-b \vee (a \vee b))$$
$$= ((-a \vee a) \vee b) \wedge ((-b \vee b) \vee a) = (1 \vee b) \wedge (1 \vee a)$$
$$= 1 \wedge 1 = 1$$

$(-a \wedge -b) \wedge (a \vee b) = (-a \wedge -b \wedge a) \vee (-a \wedge -b \wedge b)$
$= (0 \wedge -b) \vee (-a \wedge 0) = 0 \vee 0 = 0.$

SO BY UNIQUENESS OF COMPLEMENT
$-(a \vee b) = -a \wedge -b$. BY DUALITY
$-(a \wedge b) = -a \vee -b$.

(b) SUPPOSE $a \alpha b$ AND $a' \alpha b'$.
THEN $a \wedge -b,\ b \wedge -a,\ a' \wedge -b',\ b' \wedge -a' \in I$

$(a \vee a') \wedge -(b \vee b') = (a \vee a') \wedge (-b \wedge -b') =$
$((a \wedge -b) \wedge -b') \vee ((a' \wedge -b') \wedge -b)$.

$$\left.\begin{array}{l} (a \wedge -b) \wedge -b' \leq a \wedge -b \in I \Rightarrow (a \wedge -b) \wedge -b' \in I \\ (a' \wedge -b') \wedge -b \leq a' \wedge -b' \in I \Rightarrow (a' \wedge -b') \wedge -b \in I \end{array}\right\}$$

$\Rightarrow (a \vee a') \wedge -(b \vee b') \in I$

BY SYMMETRY $(b \vee b') \wedge -(a \vee a') \in I$,
SO $(a \vee a')\ \alpha\ (b \vee b')$.

$(a \wedge a') \wedge -(b \wedge b') = (a \wedge a') \wedge (-b \vee -b') =$
$((a \wedge -b) \wedge a') \vee ((a' \wedge -b') \wedge a) \in I$
SO $(a \wedge a')\ \alpha\ (b \wedge b')$

$-a \wedge (--b) = -a \wedge b \in I$
$-b \wedge (--a) = -b \wedge a \in I$
SO $-a\ \alpha\ -b$. $\alpha \in Co(\underline{B})$.

$0/\alpha = \{a \in B : a \wedge -0,\ 0 \wedge -a \in I\}$
$= \{a \in B : a, 0 \in I\} = I$

(c) $a \alpha 0$ AND $b \le a \Rightarrow$

$b = (b \wedge a) \alpha (b \wedge 0) = 0$

SO $b \le a \in 0/\alpha \Rightarrow b \in 0/\alpha$

$a \alpha 0$ AND $b \alpha 0 \Rightarrow (a \vee b) \alpha (0 \vee 0) = 0$

SO $a, b \in \underline{I} \Rightarrow a \vee b \in \underline{I}$.

HENCE $\underline{I}$ IS AN IDEAL.

$a \alpha b \Rightarrow (a \wedge -b) \alpha (b \wedge -b) = 0$

$a \alpha b \Rightarrow 0 = (a \wedge -a) \alpha (b \wedge -a)$

SO $a \alpha b \Rightarrow a \wedge -b,\ b \wedge -a \in 0/\alpha$.

CONVERSELY, $a \wedge -b,\ b \wedge -a \in 0/\alpha \Rightarrow$

$(a \wedge -b) \alpha 0,\ (b \wedge -a) \alpha 0$

$(a \wedge -b) \alpha 0 \Rightarrow a \vee b = (a \vee b) \wedge (-b \vee b) = ((a \wedge -b) \vee b) \alpha (0 \vee b) = b$. SIMILARLY

$(b \wedge -a) \alpha 0 \Rightarrow (a \vee b) \alpha a$ SO

$a \alpha b$.