Math 588, Fall 2001 **Problem Set 3** **Due Oct 23, 2001**

1. [Burris-Sanka. II.3.1] Let $\boldsymbol{A}$ be a Σ-algebra and $X \subseteq A$. Define a infinite sequence $E_0(X) \subseteq E_1(X) \subseteq E_2(X) \subseteq \cdots \subseteq A$ be recursion as follows. $E_0(X) = X$ and $E_{n+1}(X) = E_n(X) \cup \{\, \sigma^{\boldsymbol{A}}(a_1, \dots, a_m) : m \in \omega, \sigma \in \Sigma_m, a_1, \dots, a_m \in E_n(X) \,\}$. Prove that $\mathrm{Sg}^{\boldsymbol{A}}(X) = \bigcup_{n\in\omega} E_n(X)$.

2. Let Σ be the signature of groupoid, i.e., a single binary operation. Consider the binary relations of subalgebra ($\subseteq$) and homomorphic image ($\preccurlyeq$) on the class of all Σ-algebras $\mathsf{Alg}(\Sigma)$. Prove that the $\subseteq ; \succcurlyeq \; = \; \succcurlyeq ; \subseteq$, i.e., prove that for all $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{Alg}(\Sigma)$, if there exists a $\boldsymbol{C} \in \mathsf{Alg}(\Sigma)$ such that $\boldsymbol{A} \subseteq \boldsymbol{C} \succcurlyeq \boldsymbol{B}$, then there exists a $\boldsymbol{D} \in \mathsf{Alg}(\Sigma)$ such that $\boldsymbol{A} \succcurlyeq \boldsymbol{D} \subseteq \boldsymbol{B}$, and vice versa.

   [*Hint*: The "vice versa" part is the harder to prove. Under the assumptions that $\boldsymbol{D} \subseteq \boldsymbol{B}$ and there exists an epimorphism $h\colon \boldsymbol{A} \twoheadrightarrow \boldsymbol{D}$, you have to construct a "superalgebra" $\boldsymbol{C}$ of $\boldsymbol{A}$ (i.e., $\boldsymbol{A} \subseteq \boldsymbol{C}$) and an epimorphism $g\colon \boldsymbol{C} \twoheadrightarrow \boldsymbol{B}$. It is helpful to draw pictures.

   For simplicity you can assume that in this case Σ is a groupoid signature, i.e., a single binary operation (written in infix notation). Without loss of generality we assume that A and B are disjoint (otherwise $\boldsymbol{B}$ may first be replaced with an isomorphic image $\boldsymbol{B}'$ and then at the end the epimorphism $g\colon \boldsymbol{C} \twoheadrightarrow \boldsymbol{B}'$ can be composed with the isomorphism from $\boldsymbol{B}'$ to $\boldsymbol{B}$). Let $C = A \cup (B \setminus D)$. Define $g\colon C \to B$ so that $g(c) = h(c)$ if $c \in A$ and $g(c) = c$ if $c \in B \setminus D$. Then define the operation $\cdot^{\boldsymbol{C}}$ on C so that it agrees with $\cdot^{\boldsymbol{A}}$ on A and the map g is a homomorphism from $\boldsymbol{C}$ to $\boldsymbol{B}$. The definition of $c \cdot^{\boldsymbol{C}} c'$ will require the consideration of several cases depending on whether or not c and c' are in A.]

3. Let $\boldsymbol{A}$ be a groupoid. define a binary operation on the set A^A of all mappings of A into itself as follows. For all $f, g \in A^A$, $f \cdot g$ is the mapping from A to itself such that, for all $a \in A$, $(f \cdot g)(a) = f(a) \cdot^{\boldsymbol{A}} g(a)$. Prove that the set $\mathrm{End}(\boldsymbol{A})$ of endomorphisms of $\boldsymbol{A}$ is closed under this operation iff $\boldsymbol{A}$ satisfies the following *entropic law*. $(x \cdot y) \cdot (z \cdot w) \approx (x \cdot z) \cdot (y \cdot w)$. Prove that if $\boldsymbol{A}$ has an identity element (i.e., an element e such that $e \cdot^{\boldsymbol{A}} a = a = a \cdot^{\boldsymbol{A}} e$ for all $a \in A$), then $\mathrm{End}(\boldsymbol{A})$ is closed under $\cdot$ iff $\boldsymbol{A}$ is a commutative semigroup, i.e., satisfies the commutative law $x \cdot y \approx y \cdot x$ and the associative law $(x \cdot y) \cdot z \approx x \cdot (y \cdot z)$.

4. A semigroup with identity $\langle A, \cdot, e\rangle$ is called a *monoid*. Prove that every cyclic monoid is commutative. Prove that $\mathbf{H}(\langle \omega, +, 0\rangle)$, the class of all homomorphic images of $\langle \omega, +, 0\rangle$, is the class of all cyclic (commutative) monoids. Use this result and the First Isomorphism Theorem to obtain a characterization of $\mathrm{Co}(\langle \omega, +, 0\rangle)$.

   [*hint*: Show that for every monoid $\boldsymbol{A} = \langle A, \cdot, e\rangle$ and every $a \in A$, there exists a (unique) homomorphism $h\colon \langle w, +, 0\rangle \to \boldsymbol{A}$ such that $h(1) = e$.]

(over)

5. A *Boolean algebra* is a algebra $\boldsymbol{B} = \langle B, \vee, \wedge, -, 0, 1\rangle$ such that $\langle B, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice and $-$ is the *complement* operation, i.e., $\boldsymbol{B}$ satisfies the identities $-x \vee x \approx 1$ and $-x \wedge x \approx 0$.

   (a) Prove that the complement of an element is unique, i.e., if $b \vee a = 1$ and $b \vee a = 0$, then $b = -a$. Prove the *law of double negation* $--x \approx x$, and the two *DeMorgan laws*: $-(x \vee y) \approx -x \wedge -y$ and $-(x \wedge y) \approx -x \vee -y)$.

   (b) Let I be an ideal of $\boldsymbol{B}$ (in the sense of Problem #4 of Problem Set 1), and define a binary relation α on B by $a \,\alpha\, b$ if $a - b, b - a \in I$, equivalently (since I is and ideal), if $(a - b) \vee (b - a) \in I$. Prove that α is a congruence of $\boldsymbol{B}$ and that $0/\alpha = I$.

   (c) Let α be any congruence $\boldsymbol{B}$. Prove that $0/\alpha = \{\, b \in B : b \,\alpha\, a \,\}$ is an ideal of $\boldsymbol{B}$, and that, for all $a, b \in B$, $a \,\alpha\, b$ iff $a - b, b - a \in 0/\alpha$.

   Thus there is a bijection between the ideals and congruences of $\boldsymbol{B}$ that clearly preserves $\subseteq$.