SOLUTIONS PROBLEM SET #2

#1 WE FIRST SHOW THAT FOR EACH COMPACT $w \in L$

(*) $w \leq a \wedge \bigvee D$ IFF $w \leq \bigvee_{d \in D}(a \wedge d)$.

SUPPOSE $w \leq a \wedge \bigvee D$. THEN $w \leq a$ AND $w \leq \bigvee D$. THUS SINCE w IS COMPACT, $w \leq d_1 \vee \dots \vee d_m$ FOR $d_1, \dots, d_m \in D$. LET $e \in D$ BE AN UPPER BOUND OF $d_1, \dots, d_m$. (e EXISTS SINCE D IS DIRECTED). THEN $w \leq e$. SO $w \leq a \wedge e \leq \bigvee_{d \in D} a \wedge d$.

ASSUME CONVERSELY THAT $w \leq \bigvee_{d \in D}(a \wedge d)$. SINCE w IS COMPACT, $w \leq (a \wedge d_1) \vee \dots \vee (a \wedge d_m)$. LET $e \in D$ BE AN UPPER BOUND OF $d_1, \dots, d_m$. THEN $a \wedge d_i \leq a \wedge e$ FOR EACH $i \leq m$. THUS $w \leq (a \wedge d_1) \vee \dots \vee (a \wedge d_m) \leq a \wedge e$. SO $w \leq \bigvee_{d \in D}(a \wedge d)$.

SO (*) HOLDS FOR EACH COMPACT $w \in L$.

SINCE L IS ALGEBRAIC,

$$\begin{aligned} a \wedge \bigvee D &= \bigvee \{w \in \mathrm{Comp}(\underline{L}) : w \leq a \wedge \bigvee D\} \\ &= \bigvee \{w \in \mathrm{Comp}(\underline{L}) : w \leq \bigvee_{d \in D}(a \wedge d)\} \\ &= \bigvee_{d \in D}(a \wedge d). \end{aligned}$$

#2 WE FIRST PROVE THAT

(*) $a \in Sg^{\underline{A}}(Y)$ IMPLIES $\exists Y' \subseteq_\omega Y \, (a \in Sg^{\underline{A}}(Y'))$

BY #1 ON PROB SET #3

$Sg^{\underline{A}}(Y) = \bigcup_{n \in \omega} E_n(Y)$. So $a \in E_n(Y)$ FOR SOME $n \in \omega$. WE PROVE THAT

(**) $\exists Y' \subseteq_\omega Y \, (a \in Sg^{\underline{A}}(Y'))$ BY INDUCTION ON n.

IF $n = 0$, $a \in Y$ AND CAN TAKE $Y' = \{a\}$.

ASSUME $a \in E_{n+1}(Y)$. THEN EITHER $a \in E_n(Y)$ AND (**) HOLDS BY IND. HYP. OR $a = \sigma^{\underline{A}}(b_1, \ldots, b_m)$ WITH $b_1, \ldots, b_m \in E_n(Y)$. BY IND. HYP. $\exists Y_1', \ldots, Y_m' \subseteq_\omega Y$ SUCH THAT $b_1 \in Sg^{\underline{A}}(Y_1), \ldots, b_m \in Sg^{\underline{A}}(Y_m)$. TAKE $Y' = Y_1' \cup \cdots \cup Y_m'$. THEN (**) HOLDS.

SO (*) HOLDS.

SUPPOSE NOW THAT $A = Sg^{\underline{A}}(Y)$ AND ALSO $A = Sg^{\underline{A}}(X)$ FOR SOME FINITE $X \subseteq_\omega A$. LET $X = \{x_1, \ldots, x_n\}$. THEN BY (*) $\exists Y_1', \ldots, Y_n' \subseteq_\omega Y$ SUCH THAT $x_1 \in Sg^{\underline{A}}(Y_1'), \ldots, x_n \in Sg^{\underline{A}}(Y_n')$. LET $Y' = Y_1' \cup \cdots \cup Y_n'$. $X \subseteq Sg^{\underline{A}}(Y')$.

HENCE $A = Sg^{\underline{A}}(X) = Sg^{\underline{A}}(Y')$.

#3 a) LET $E \in Eq(A)_\omega$ SUCH THAT E IS GENERATED BY A FINITE SET OF ORDERED PAIRS. FOR EACH $a \in A$, $b \in [a]_E$ IFF $\exists$ $\langle x_1, x_2\rangle, \langle x_2, x_3\rangle, \ldots, \langle x_{m-2}, x_{m-1}\rangle, \langle x_{m-1}, x_m\rangle$ SUCH THAT $x_1 = a$, $x_m = b$, AND $\forall i < m\,(\langle x_i, x_{i+1}\rangle \in X$ OR $\langle x_{i+1}, x_i\rangle \in X)$

SINCE X IS FINITE, EACH $[a]_E$ IS FINITE AND $[a]_E \neq \{a\}$ FOR ONLY A FINITE NUMBER OF $a \in A$. I.E., THE PARTITION $P(E)$ OF E HAS THE FOLLOWING PROPERTY:

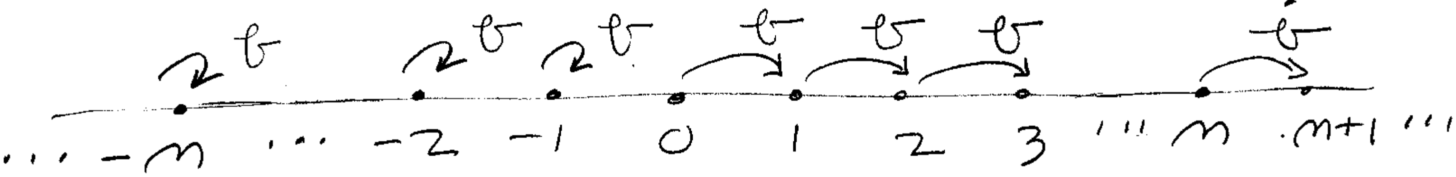
* ($P(E)$ CONTAINS A FINITE NUMBER OF NON-SINGLETON EQUIVALENCE CLASSES AND EACH OF THEM IS FINITE

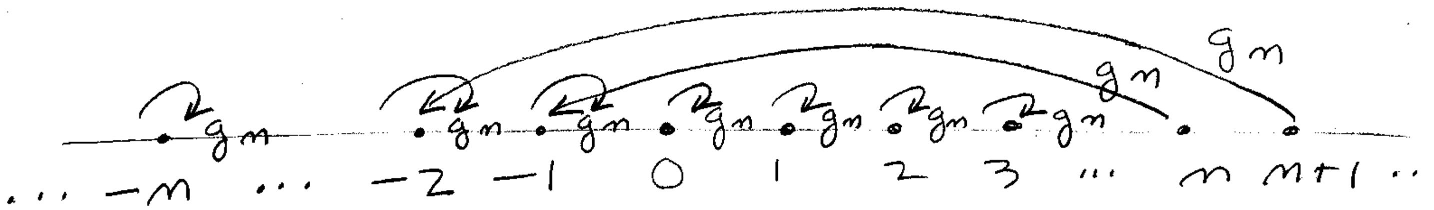
CONVERSELY, EACH EQUIVALENCE RELATION WITH THIS PROPERTY IS FINITELY GENERATED.

SUPPOSE NOW THAT $E, F \in Eq(A)$, E IS FINITELY GENERATED, AND $F \subseteq E$. $P(E)$ HAS PROPERTY (*). SINCE $F \subseteq E$, FOR EACH $a \in A$ WE HAVE $[a]_F \subseteq [a]_E$. SO $P(F)$ HAS PROPERTY (*) AND HENCE F IS FINITELY GENERATED. SUPPOSE THAT $\mathcal{K} \subseteq Eq(A)_\omega$. IF $\mathcal{K} \neq \emptyset$, THEN $\bigcap \mathcal{K} \subseteq E$ WHERE E IS ANY MEMBER OF

$\mathcal{K}$. SO $\bigcap \mathcal{K} \in Eq(A)_\omega$. HOWEVER, IF $\mathcal{K} = \emptyset$, THEN $\bigcap \mathcal{K} = A \times A$ WHICH IS CLEARLY NOT FINITELY GENERATED IF A IS INFINITE. SO CONTRARY TO WHAT IS CLAIMED IN STATEMENT OF PROBLEM, $Eq(A)_\omega$ IS NOT A CLOSED SET SYSTEM IF A IS INFINITE. IT COMES VERY CLOSE HOWEVER.

(b) FOR EACH $n \in \omega$ LET $g_n(x) = g(n, x)$ THE FOLLOWING IS A GRAPHICAL REPRESENTATION OF THE ALGEBRA $\underline{A}$.



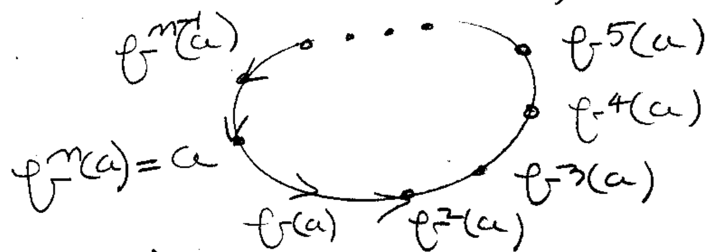


$$Sg^{\underline{A}}(\{n\}) = (-1] \cup [n) \quad \forall n \in \omega$$

SO $\bigcap_{n \in \omega} Sg^{\underline{A}}(n) = (-1]$

BUT $(-1]$ IS NOT GENERATED BY ANY PROPER SUBSET AND HENCE IS NOT FINITELY GENERATED.

#4(a) LET $\underset{\frown}{A} = \langle A, f \rangle$ BE A MONO-UNARY ALGEBRA AND ASSUME $\underset{\frown}{A}$ HAS NO PROPER SUBUNIVERSE. WE PROVE IT MUST BE A CYCLE, IE., OF THE FORM AND HENCE FINITE.



LET $a \in A$. $A = Sg^{\underset{\frown}{A}}(\{a\})$ SINCE OTHERWISE $Sg^{\underset{\frown}{A}}(\{a\})$ WOULD BE A PROPER SUBUNIVERSE. SO $\underset{\frown}{A}$ IS CYCLIC. IF ITS TAILLENGTH IS POSITIVE THEN $Sg^{\underset{\frown}{A}}(\{f(a)\})$ IS A PROPER SUBUNIVERSE SO $\underset{\frown}{A}$ MUST BE A CYCLE.

(b) LET $\underset{\frown}{A} = \langle \omega, S, P \rangle$ WHERE S IS THE SUCCESSOR FUNCTION AND P IS THE PREDECESSOR. $Sg^{\underset{\frown}{A}}(\{n\}) = \omega$ FOR EVERY $n \in \omega$. SO $\underset{\frown}{A}$ HAS NO PROPER SUBUNIVERSE. (THE EMPTY UNIVERSE IS NOT PROPER).

#5 LET $K \in \mathrm{Sub}(\langle \mathbb{Z}, +, 0\rangle)$. THEN $K \cap \omega \in \mathrm{Sub}(\langle \omega, +, 0\rangle)$. $(0] = \{a \in \mathbb{Z} : a \leq 0\}$ IS A SUBUNIVERSE OF $\langle \mathbb{Z}, +, 0\rangle$ AND THE SUBALGEBRA $\langle (0], +, 0\rangle$ OF $\langle \mathbb{Z}, +, 0\rangle$ IS ISOMORPHIC TO $\langle \omega, +, 0\rangle$ UNDER THE MAP $a \mapsto -a$

SUPPOSE EVERY SUBUNIVERSE OF $\langle \omega, +, 0\rangle$ IS FINITELY GENERATED. THEN BY THE ISOMORPHISM EVERY SUBUNIVERSE OF $\langle (0], +, 0\rangle$ IS FINITELY GENERATED. LET $K \in \mathrm{Sub}(\langle \mathbb{Z}, +, 0\rangle)$. THEN $K \cap \omega$ AND $K \cap (0]$ ARE SUBUNIVERSES OF $\langle \omega, +, 0\rangle$ AND $\langle (0], +, 0\rangle$ RESPECTIVELY. LET $X \subseteq_\omega \omega$ AND $Y \subseteq_\omega (0]$ SUCH THAT

$K \cap \omega = \mathrm{Sg}^{(\langle \omega, +, 0\rangle)}(X)$ AND

$K \cap (0] = \mathrm{Sg}^{(\langle (0], +, 0\rangle)}(Y)$.

CLEARLY $\mathrm{Sg}^{(\langle \omega, +, 0\rangle)}(X) \subseteq \mathrm{Sg}^{(\langle \mathbb{Z}, +, 0\rangle)}(X)$ AND $\mathrm{Sg}^{(\langle (0], +, 0\rangle)}(Y) \subseteq \mathrm{Sg}^{(\langle \mathbb{Z}, +, 0\rangle)}(Y)$. SO $K \subseteq \mathrm{Sg}^{(\langle \mathbb{Z}, +, 0\rangle)}(X \cup Y)$. BUT SINCE $X \cup Y \subseteq K$ AND $K \in \mathrm{Sub}(\langle \mathbb{Z}, +, 0\rangle)$, THE INCLUSION IN THE OPPOSITE DIRECTION ALSO HOLDS.

SO IF EVERY SUBUNIVERSE OF $\langle \omega, +, 0\rangle$ IS FINITELY GENERATED, SO IS EVERY SUBUNIVERSE OF $\langle \mathbb{Z}, +, 0\rangle$. CONSEQUENTLY

WE NEED CONSIDER ONLY SUBUNIVERSES OF $\langle \omega, +, 0 \rangle$.

LET $K \in \mathrm{SUB}(\langle \omega, +, 0 \rangle)$. CAN ASSUME $K \neq \{0\}$. LET $K^+ = \{a \in K : a > 0\}$. LET $M = \mathrm{GCD}(K^+)$. THERE IS A FINITE SET $k_1, \ldots, k_\ell$ OF ELEMENTS OF K^+ SUCH THAT $M = \mathrm{GCD}(k_1, \ldots, k_\ell)$. THERE EXIST $u_1, \ldots, u_\ell \in \mathbb{Z} \sim \{0\}$ SUCH THAT $M = u_1 k_1 + \cdots + u_\ell k_\ell$. LET $a_1, \ldots, a_s$ BE THOSE k_i SUCH THAT $u_i > 0$ AND $b_1, \ldots, b_t$ THOSE k_i SUCH THAT $u_i < 0$. THEN WE HAVE

$$M = (p_1 a_1 + \cdots + p_s a_s) - (q_1 b_1 + \cdots + q_t b_t)$$

WHERE $a_1, \ldots, a_s, b_1, \ldots, b_t \in K^+$, AND $p_1, \ldots, p_s, q_1, \ldots, q_t \in \omega \sim \{0\}$.

LET $m = \left(\sum q_j b_j\right)^2$; CLEARLY $m \in K$.

<u>CLAIM</u> $K \cap [m) = \{m + kM : k \in \omega\}$.

PROOF OF CLAIM.

$\subseteq$: LET $u \in K \cap [m)$, IE., $u \in K$ AND $m \leq u$. u AND m ARE BOTH DIVISIBLE BY M SINCE $M = \mathrm{GCD}(K^+)$. SO $u - m$ IS DIVISIBLE BY M AND HENCE $u = m + kM$, AND $k \geqslant 0$ SINCE $m \leq u$.

$\supseteq$: WE MUST SHOW $m + hm \in K$ FOR ALL $h \in \omega$. ASSUME FIRST OF ALL THAT $h \le \sum q_j b_j$.

$$m + hm = \left(\sum q_j b_j\right)^2 + h\left(\sum p_i a_i - \sum q_i b_j\right)$$
$$= \left(\sum q_j b_j\right)\underbrace{\left(\sum q_j b_j - h\right)}_{\ge 0} + h \sum p_i a_i \in K$$

ASSUME NOW THAT $h > \sum q_j b_j$. THEN BY THE DIVISION ALGORITHM

$$h = s\left(\sum q_j b_j\right) + v \quad \text{WITH } s > 0$$
$$\text{AND } 0 \le v < \sum q_j b_j.$$

$$m + hm = m + \left(s\left(\sum q_j b_j\right) + v\right) m$$
$$= m + vm + (sm)\left(\sum q_j b_j\right)$$

$m + vm \in K$ BY FIRST CASE AND $(sm)\left(\sum q_j b_j\right) \in K$ SINCE $sm > 0$.

$\boxed{X}$ CLAIM

IT REMAINS ONLY TO SHOW THAT K IS FINITELY GENERATED. SINCE IN GENERAL $m \notin K$ THIS REQUIRES A LITTLE WORK. LET

$$X = (K \cap [1, m)) \cup \{m + hm : 0 \le h < m\}$$

<u>CLAIM</u> $K = Sg(\langle \omega, +, 0 \rangle)(X)$

PROOF OF CLAIM. THE INCLUSION $\supseteq$ IS CLEAR.

CONSIDER ANY $a \in K$. IF $a < m$ THEN $a \in X \subseteq Sg^{\langle \omega, +, 0 \rangle}(X)$. OTHERWISE $a = m + km$ FOR SOME $k \in \omega$ BY FIRST CLAIM. LET $k = lm + u$ WITH $l > 0$ AND $0 \leq u < m$. THEN $a = m + (lm + u)m = (lm)m + (m + um)$. $(lm)m \in Sg^{\langle \omega, +, 0 \rangle}(X)$ SINCE $m \in X$ AND $m + um \in X$. SO $a \in Sg^{\langle \omega, +, 0 \rangle}(X)$.