Math 588, Fall 2001 **Hints for Problem Set 2**

Problem 1 For any element a in an algebraic lattice $\boldsymbol{L}$, $a = \bigvee\{\, c \in \mathrm{Comp}(\boldsymbol{L}) : c \leq a \,\}$. So in order to prove that two elements are equal it suffices to prove that they bound above exactly the same set of compact elements. Assume D is upward directed and $c \in \mathrm{Comp}(\boldsymbol{L})$. Let $c \leq a \wedge \bigvee D$. Then $c \leq a$ and $c \leq \bigvee D$. Then $c \leq \bigvee D'$ for some $D' \subseteq_\omega D$ this c is compact and hence $c \leq e$ for some $e \in D$ (why?). So $c \leq a \wedge e$ and hence $c \leq \bigvee_{d \in D}(a \wedge d)$. The implication in the other direction is left to you.

Problem 2 I see that I should have had you first prove the following result. Let $\boldsymbol{A}$ be a Σ-algebra and $Y \subseteq A$. We define a infinite sequence $E_0 \subseteq E_1 \subseteq E_2 \subseteq \cdots \subseteq A$ be recursion as follows. $E_0 = Y$ and $E_{n+1} = \{\, \sigma^{\boldsymbol{A}}(a_1, \ldots, a_n) : n \in \omega, \sigma \in \Sigma_n, a_1, \ldots, a_n \in E_n \,\}$. Then

$$\mathrm{Sg}^{\boldsymbol{A}}(Y) = \bigcup_{n \in \omega} E_n. \tag{1}$$

I'm putting this on the next problem set. You can assume it here.

First show that in order to prove the result in Problem 2 it suffices to show that for all $a \in A$,

$$a \in \mathrm{Sg}^{\boldsymbol{A}}(Y) \quad \text{implies} \quad \text{there is a } Y' \subseteq_\omega Y \text{ such that } a \in \mathrm{Sg}^{\boldsymbol{A}}(Y') \tag{2}$$

In order to prove this, use (1) to prove that following by induction on n.

$$\text{for all } n \in \omega,\ a \in E_n \text{ implies there is a } Y' \subseteq_\omega Y \text{ such that } a \in \mathrm{Sg}^{\boldsymbol{A}}(Y').$$

Problem 5 Let $a, b \in \omega$ and assme neither divides the other. Then $\mathrm{GCD}(a, b) = qa + pb$ where exactly one of q and p is negative. WLOG can assume it's p. So we can assume $\mathrm{GCD}(a, b) = qa - pb$ where q and p are both positive. Let $n = \mathrm{GCD}(a, b)$. Let $m = (pb)^2$. Prove that all natural numbers of the form $m + kn$ are in $\mathrm{Sg}^{\langle \omega, + \rangle}(a, b)$, and every element of $\mathrm{Sg}^{\langle \omega, + \rangle}(a, b)$ that is $\geq m$ is of this form.