Math 588, Fall 2001 **Problem Set 2** **Due Oct, 9, 2001**

(1) [Burris-Sanka. 1.4.6] Let $\boldsymbol{L} = \langle L, \leq \rangle$ be an algebraic lattice and D an upward directed subset of L, i.e., for all $d_1, d_2 \in D$ there exists a $d_3 \in D$ such that $d_1 \leq d_3$ and $d_2 \leq d_3$. Prove that, for every $a \in L$, $a \wedge \bigvee D = \bigvee_{d \in D}(a \wedge d)$.
[*Hint*: Show that, for every compact element c of $\boldsymbol{L}$, $c \leq a \wedge \bigvee D$ iff $c \leq \bigvee_{d \in D}(a \wedge d)$.]

(2) [based on Burris-Sanka. 1.5.7] Let Σ be an arbitrary signature and let $\boldsymbol{A} = \langle A, \{\sigma^A : \sigma \in \Sigma\}\rangle$ be a finitely generated Σ-algebra, i.e., there exists a finite $X \subseteq A$ such that $A = \mathrm{Sg}^{\boldsymbol{A}}(X)$. Prove that, for every set Y that generates $\boldsymbol{A}$, there is a finite $Y' \subseteq Y$ such that Y' generates $\boldsymbol{A}$.

(3) Let $\langle A, \mathcal{C}\rangle$ be a closed-set system, and let $\mathcal{C}_\omega$ be the set of all finitely generated closed sets, i.e., $\mathcal{C}_\omega = \{\,\mathrm{Cl}_{\mathcal{C}}(X) : X \subseteq_\omega A\,\}$. $\langle A, \mathcal{C}_\omega\rangle$ may or may not be a closed-set system, i.e., the set of finitely generated sets may or may not be closed under intersection.
   (a) Prove that, for any set A, $\langle A, \mathrm{Eq}(A)_\omega\rangle$, i.e., the set of all finitely generated equivalence relations, is a closed-set system.
   [*Hint*: Show that every subequivalence relation of a finitely generated equivalence relation is finitely generated. Look at the associated partitions.]
   (b) Let $\Sigma = \{f, g\}$ where f is unary and g is binary. Let $\boldsymbol{A} = \langle \mathbb{Z}, f, g\rangle$ be the Σ-algebra whose universe is the set of integers and f and g are defined as follows. $f(n) = n + 1$ if $0 \leq n$; otherwise $f(n) = n$. $g(n, m) = n - m$ if $0 \leq n < m$; otherwise $g(n, m) = n$. ("$+$", "$-$", and "$\leq$" are the usual addition, substraction, and order of the additive group of integers.) Show that $\langle \mathbb{Z}, \mathrm{Sub}(\boldsymbol{A})_\omega\rangle$ is not a closed-set system.

(4) (a) Prove that every infinite mono-unary algebra has a proper subuniverse (i.e, a subuniverse different from both $\emptyset$ and the universe of the algebra).
   (b) Construct an infinite *bi-unary algebra* (i.e., two unary operations) that has no proper subuniverse.

(5) The subuniverses of $\langle \mathbb{Z}, +\rangle$ different from $\langle \mathbb{Z}, +, -, 0\rangle$ and are much harder to characterize. Show that they are all finitely generated.
[*Hint*: Show that the problem can be reduced to showing that every subuniverse of $\langle \omega, +\rangle$ is finitely generated. A subuniverse A of $\langle \omega, +\rangle$ is *eventually periodic* if there exists $n, m \in \omega$ with $m > 0$ such that every $x \in A$ with $x \geq m$ is of the form $m + kn$ with $k \in \omega$. For example, if A is the subalgebra of $\langle \omega, +\rangle$ generated by 3 and 5, then A is eventually periodic with $m = 8$ and $n = 1$. Prove that every subuniverse of $\langle \omega, +\rangle$ is eventually periodic. Use this fact to prove that every subuniverse is finitely generated.]