#1. LET $\leq'$ BE A MAXIMAL PARTIAL ORDERING OF A SUCH THAT $\leq \subseteq \leq'$, IE, $\langle A, \leq' \rangle$ IS A POSET, $\forall a, b \in A (a \leq b \Rightarrow a \leq' b)$, AND $\forall R$ SUCH THAT $\leq' \subset R \subseteq A^2$, $\langle A, R \rangle$ IS NOT A POSET. CLEARLY SUCH A $\leq'$ EXISTS BECAUSE A IS FINITE. THE CLAIM IS THAT $\leq'$ IS A LINEAR ORDERING. SUPPOSE $\exists a, b \in A$ SUCH THAT $a \not\leq' b$ AND $b \not\leq' a$. LET $R = \leq' \cup \{\langle x, y \rangle \in A^2 : x \leq' a \text{ AND } b \leq' y\}$. $\langle a, b \rangle \in R$ SINCE $a \leq' a$ AND $b \leq' b$.

<u>CLAIM</u>: R IS A PARTIAL ORDERING.

$\Delta_A \subseteq R$ SINCE $\Delta_A \subseteq \leq'$. SUPPOSE $\langle c, d \rangle, \langle d, e \rangle \in R$. IF $\langle c, d \rangle, \langle d, e \rangle \in \leq'$, THEN $\langle c, e \rangle \in \leq' \subseteq R$. SUPPOSE $\langle c, d \rangle \in \{\langle x y \rangle : x \leq' a \text{ AND } b \leq' y\}$, IE, $c \leq' a$, $b \leq' d$. THEN $\langle d, e \rangle \notin \{\langle x, y \rangle : x \leq' a \text{ AND } b \leq' y\}$ SINCE OTHERWISE $b \leq' d$ AND $d \leq' a$ IMPLIES $b \leq' a$, WHICH IS FALSE. SO $d \leq' e$, AND $c \leq' a$, $b \leq' e$, IE. $\langle c, e \rangle \in \{\langle x, y \rangle : x \leq' a, b \leq' y\} \subseteq R$. SIMILARLY, IF $\langle d, e \rangle \in \{\langle x, y \rangle : x \leq' a \text{ AND } b \leq' y\}$ THEN $c \leq' d$ AND HENCE $c \leq' a$ AND $b \leq' e$, SO AGAIN $\langle c, e \rangle \in R$. $\langle c, d \rangle, \langle d, e \rangle$ CANNOT BOTH BE IN $\{\langle x, y \rangle : x \leq' a \text{ AND } b \leq' y\}$

SINCE $c, d \le' a$ AND $b \le' d, e$ IMPLY $b \le' a$. SO R IS TRANSITIVE.
TO SEE IT'S ANTISYMMETRIC, SUPPOSE $\langle c, d\rangle, \langle d, c\rangle \in R$. IF $c \le' d$ AND $d \le' c$ THEN $c = d$. SUPPOSE $c \le' a$ AND $b \le' d$. IF $d \le' c$, THEN $b \le' d \le' c \le' a$, FALSE. SO $d \le' a$ AND $b \le' c$, BUT THEN $b \le' c \le' d \le' a$, AGAIN FALSE. SO R IS ANTISYMMETRIC.

$\boxed{\times}$ CLAIM

SO R IS A PARTIAL ORDERING THAT PROPERLY INCLUDES $\le'$. THIS CONTRADICTS THE MAXIMALITY OF $\le'$. SO $\le'$ IS A LINEAR ORDERING.

#2. SUPPOSE NOT EVERY ELEMENT OF A IS THE JOIN OF A FINITE NUMBER OF JOIN IRREDUCIBLES. LET a BE A MINIMAL ELEMENT (UNDER THE LATTICE ORDERING) OF THIS KIND. SUCH AN ELEMENT EXISTS SINCE A IS FINITE. IN PARTICULAR a ITSELF IS NOT JOIN IRREDUCIBLE. SO $a = c \vee d$ SUCH THAT $c < a$ AND $d < a$. BY THE MINIMALITY OF a, $c = c_1 \vee \dots \vee c_n$ AND $d = d_1 \vee \dots \vee d_m$ WHERE THE c_i AND d_j ARE JI. THEN $a = c_1 \vee \dots \vee c_n \vee d_1 \vee \dots \vee d_m$ IS A REPRESENTATION OF a AS A FINITE JOIN OF JI'S. CONTRADICTION.

#3 SUPPOSE S AND T ARE LOWER SEGMENTS. SUPPOSE $a \leq u \in S \cap T$. THEN $a \in S$ AND $a \in T$, I.E., $a \in S \cap T$. SO $S \cap T$ IS A LOWER SEGMENT.

IF R IS A LOWER SEGMENT SUCH THAT $R \subseteq S$ AND $R \subseteq T$, THEN $R \subseteq S \cap T$. SO $S \cap T = GLB(S, T)$ UNDER $\subseteq$.

SUPPOSE NOW THAT $a \leq u \in S \cup T$. THEN EITHER $a \leq u \in S$ OR $a \leq u \in T$. SO $a \in S$ OR $a \in T$, I.E., $a \in S \cup T$. SO $S \cup T$ IS A LOWER SEGMENT.

IF R IS A LOWER SEGMENT SUCH THAT $S \subseteq R$ AND $T \subseteq R$, THEN $S \cup T \subseteq R$. SO $S \cup T = LUB(S, T)$ UNDER $\subseteq$.

$\langle L(A) \cup \{\emptyset\}, \cup, \cap \rangle$ IS A LATTICE.

IF A HAS A LEAST ELEMENT 0, THEN 0 IS IN EVERY NONEMPTY LOWER SEGMENT AND $L(A)$ IS CLOSED UNDER INTERSECTION (AND OBVIOUSLY UNION). SO $\langle L(A), \cup, \cap \rangle$ IS A SUBLATTICE OF $\langle L(A) \cup \{\emptyset\}, \cup, \cap \rangle$.

\#4 WE FIRST SHOW THAT $I(\underset{\sim}{A}) \cup \{\emptyset\}$ CLOSED UNDER ARBITRARY INTERSECTION. LET $\mathcal{K} \subseteq I(\underset{\sim}{A})$. IF $u \leq w \in \bigcap \mathcal{K}$. THEN $u \leq w \in K$ FOR EACH $K \in \mathcal{K}$. SO $\forall K \in \mathcal{K}(u \in K)$ AND HENCE $u \in \bigcap \mathcal{K}$. SO $\bigcap \mathcal{K}$ IS A LOWER SEGMENT.

$a, b \in \bigcap \mathcal{K} \Rightarrow \forall K \in \mathcal{K}(a, b \in K) \Rightarrow \forall K \in \mathcal{K}(a \vee b \in K) \Rightarrow a \vee b \in \bigcap \mathcal{K}$.

SO $\bigcap \mathcal{K} \in I(\underset{\sim}{A}) \cup \{\emptyset\}$. SO $I(\underset{\sim}{A}) \cup \{\emptyset\}$ IS A CLOSED-SET SYSTEM. HENCE $\langle I(\underset{\sim}{A}) \cup \{\emptyset\}, \cap, \vee \rangle$ IS A COMPLETE LATTICE WITH $I \vee J = \bigcap \{K \in I(\underset{\sim}{A}) \cup \{\emptyset\} : I \cup J \subseteq K\}$

LET $I, J \in I(\underset{\sim}{A})$. SO $I, J \neq \emptyset$. LET $a \in I$ AND $b \in J$. THEN $a \wedge b \in I \cap J$ SO $I \cap J \neq \emptyset$. AND HENCE $I \cap J \in I(\underset{\sim}{A})$. SO $I(\underset{\sim}{A})$ IS CLOSED UNDER <u>FINITE</u> MEETS (AND OBVIOUSLY ARBITRARY JOINS) SO $\langle I(\underset{\sim}{A}), \cap, \vee \rangle$ IS A SUBLATTICE OF $\langle I(\underset{\sim}{A}) \cup \{\emptyset\}, \cap, \vee \rangle$.

<u>IN ORDER</u> TO SHOW THE 2ND PART OF THE PROBLEM WE FIRST SHOW THAT

(*) $I \vee J = \{a \in A : \exists i \in I, j \in J \, (a \leq i \vee j)\}$

LET L BE THE SET ON THE RIGHT.

FOR ANY $i \in I$ AND $j \in J$, $i \leq i \vee j$ AND $j \leq i \vee j$. SO $I \subseteq L$ AND $J \subseteq L$. THUS $L \neq \emptyset$. WE SHOW IT IS AN IDEAL. SUPPOSE $s \leq a \in L$. THEN $s \leq a \leq i \vee j$. HENCE $s \in L$. SUPPOSE $a, b \in L$, IE, $a \leq i \vee j$, $b \leq i' \vee j'$. THEN $a \vee b \leq (i \vee j) \vee (i' \vee j') = \underbrace{(i \vee i')}_{I} \vee \underbrace{(j \vee j')}_{J}$. SO $a \vee b \in L$. THUS $L \in I(\underset{\sim}{A})$. LET $M \in I(\underset{\sim}{A})$ SUCH THAT $I \subseteq M$ AND $J \subseteq M$. SUPPOSE $s \leq i \vee j$, $i \in I$, $j \in J$. THEN $i, j \in M$, SO $s \leq i \vee j \in M$ AND HENCE $s \in M$. SO $L \subseteq M$. THIS GIVES $(*)$.

NOW ASSUME $\underset{\sim}{A}$ IS DISTRIBUTIVE. TO SHOW $\langle I(\underset{\sim}{A}), \vee, \wedge \rangle$ DISTRIBUTIVE IT SUFFICES TO SHOW THAT FOR ALL $I, J, K \in I(\underset{\sim}{A})$

$$I \cap (J \vee K) \subseteq (I \cap J) \vee (I \cap K)$$

(THAT IS IS SUFFICENT WAS PROVED IN CLASS). LET $a \in I \cap (J \vee K)$. THEN $a \in I$ AND $a \leq j \vee k$ WITH $j \in J$, $k \in K$. THEN $a = a \wedge (j \vee k) = \underbrace{(a \wedge j)}_{I \cap J} \vee \underbrace{(a \wedge k)}_{I \cap K}$

SO $a \in (I \cap J) \vee (I \cap K)$.

#5 LET $\mathcal{K} \subseteq \text{SUB}(\underset{\sim}{A})$.

<u>CLAIM</u> $\bigcap \mathcal{K} \in \text{SUB}(\underset{\sim}{A})$

$a, b \in \bigcap \mathcal{K} \Rightarrow \forall K \in \mathcal{K}(a, b \in K) \Rightarrow$
$\forall K \in \mathcal{K}(a \vee b, a \wedge b \in B) \Rightarrow a \vee b, a \wedge b \in \bigcap \mathcal{K}$

CLEARLY $0, 1 \in \bigcap \mathcal{K}$. SO $\text{SUB}(\underset{\sim}{A})$ FORMS A CLOSED-SET SYSTEM AND HENCE $\langle \text{SUB}(\underset{\sim}{A}), \cap, \vee \rangle$ IS A COMPLETE LATTICE WITH

$$\bigvee \mathcal{K} = \bigcap \{ B \in \text{SUB}(\underset{\sim}{A}) : \forall K \in \mathcal{K}(K \subseteq B) \}$$

LET $H, K \in \text{SUB}(\underset{\sim}{A})$ AND LET

$$L = \{ (h_1 \wedge k_1) \vee \cdots \vee (h_m \wedge k_m) : m \in \omega, \; h_1, \ldots, h_m \in H, k_1, \ldots, k_m \in K \}$$

BY INDUCTION ON m $(h_1 \wedge k_1) \vee \cdots \vee (h_m \wedge k_m) \in H \vee K$. SO $L \subseteq H \vee K$.

$h \in K \Rightarrow h = h \wedge 1 \in L$ AND $k \in K \Rightarrow k = 1 \wedge k \in L$. SO $H \subseteq L$ AND $K \subseteq L$.

$0 = 0 \wedge 0 \in L$ AND $1 = 1 \wedge 1 \in L$. TO SEE THAT $H \vee K = L$ IT REMAINS ONLY TO SHOW THAT $L \in \text{SUB}(\underset{\sim}{A})$, I.E., CLOSED UNDER $\vee$ AND $\wedge$. LET

$a = (h_1 \wedge k_1) \vee \cdots \vee (h_m \wedge k_m)$ AND

$a' = (h_1' \wedge k_1') \vee \cdots \vee (h_m' \wedge k_m')$.

$a \vee a' = (h_1 \wedge k_1) \vee \cdots \vee (h_m \wedge k_m) \vee (h_1' \vee k_1') \vee \cdots \vee (h_m' \wedge k_m') \in L$

$$a \wedge b = \left((h_1 \wedge k_1) \vee \cdots \vee (h_n \wedge k_n) \right) \wedge \left((h_1' \wedge k_1') \vee \cdots \vee (h_m' \wedge k_m') \right)$$

$$= \left(\bigvee_{i \leq n} h_i \wedge k_i \right) \wedge \left(\bigvee_{j \leq m} h_j' \wedge k_j' \right)$$

$$= \bigvee_{i \leq n} \bigvee_{j \leq m} (h_i \wedge k_i) \wedge (h_j' \wedge k_j')$$

BY DISTRIBUTIVITY

$$= \bigvee_{i \leq n} \bigvee_{j \leq m} \underbrace{(h_i \wedge h_j')}_{H} \wedge \underbrace{(k_i \wedge k_j')}_{K} \in L$$