1. [Burris-Sanka. 1.1.9] Let \( \langle A, \leq \rangle \) be a finite poset. Show that there is a total (i.e., linear) order \( \leq' \) on \( A \) such that \( \leq \subseteq \leq' \), i.e., \( a \leq b \) implies \( a \leq' b \).

Hint: consider the set of all partial orders \( \preceq \) on \( A \) such \( \leq \subseteq \preceq \). Show that there must be a maximal one and that any maximal one is a total order. The result holds also for infinite posets, but Zorn’s lemma must be used in this case.

2. [Burris-Sanka. 1.1.10] Let \( A = \langle A, \lor, \land \rangle \) be a lattice. An element \( a \in A \) is join irreducible if \( a = b \lor c \) implies \( a = b \) or \( a = c \). If \( A \) is a finite lattice, show that every element is of the form \( a_1 \lor \cdots \lor a_n \), where each \( a_i \) is join irreducible.

3. [Burris-Sanka. 1.2.4] Let \( A = \langle A, \leq \rangle \) be a poset. A subset \( S \) of \( A \) is a lower segment of \( A \) if every element of \( A \) that is less than or equal to some element of \( S \) is in \( S \), i.e., for all \( a \in A \) and \( s \in S \), \( a \leq s \) implies \( a \in S \). Show that the lower segments of \( A \) form a lattice with operations under \( \cup \) and \( \cap \) (the set-theoretical join and meet). If \( A \) has a least element, show that the set \( L(A) \) of non-empty lower segments of \( A \) forms a lattice.

4. [Burris-Sanka. 1.2.5 and 1.3.2] If \( A = \langle A, \lor, \land \rangle \) is a lattice, then an ideal of \( A \) is a nonempty lower segment that is closed under \( \lor \). Show that the set \( I(A) \) of ideals of \( A \) forms a lattice under \( \subseteq \).

If \( A \) is distributive, show that \( \langle I(A), \subseteq \rangle \) is distributive.

5. Let \( A \) be a bounded lattice (a lattice is bounded if it has a least element 0 and a greatest element 1). Let \( \text{Sub}(A) \) be the set of all sublattices of \( A \) that include 0 and 1. Show that \( \text{Sub}(A) = \langle \text{Sub}(A), \subseteq \rangle \) is a complete lattice.

Show that, if \( A \) is distributive, then for all \( H, K \in \text{Sub}(A) \), \( H \lor K \) consists of all elements of \( A \) of the form \( (h_1 \land k_1) \lor \cdots \lor (h_n \land k_n) \), with \( 1 \leq n \in \omega, h_1, \ldots, h_n \in H \) and \( k_1, \ldots, k_n \in K \).