Math 588, Fall 2001 **Problem Set 1** **Due Sept. 18, 2001**

1. [Burris-Sanka. 1.1.9] Let $\langle A, \leq \rangle$ be a be a finite poset. Show that there is a total (i.e., linear) order $\leq'$ on A such that $\leq \subseteq \leq'$, i.e., $a \leq b$ implies $a \leq' b$.

   Hint: consider the set of all partial orders $\preccurlyeq$ on A such $\leq \subseteq \preccurlyeq$. Show that there must be a maximal one and that any maximal one is a total order. The result holds also for infinite posets, but Zorn's lemma must be used in this case.]

2. [Burris-Sanka. 1.1.10] Let $\boldsymbol{A} = \langle A, \vee, \wedge \rangle$ be a lattice. An element $a \in A$ is *join irreducible* if $a = b \vee c$ implies $a = b$ or $a = c$. If $\boldsymbol{A}$ is a finite lattice, show that every element is of the form $a_1 \vee \cdots \vee a_n$, where each a_i is join irreducible.

3. [Burris-Sanka. 1.2.4] Let $\boldsymbol{A} = \langle A, \leq \rangle$ be a poset. A subset S of A is a *lower segment* of $\boldsymbol{A}$ if every element of A that is less than or equal to some element of S is in S, i.e., for all $a \in A$ and $s \in S$, $a \leq s$ implies $a \in S$. Show that the lower segments of $\boldsymbol{A}$ form a lattice with operations under $\cup$ and $\cap$ (the set-theoretical join and meet). If $\boldsymbol{A}$ has a least element, show that the set $L(\boldsymbol{A})$ of non-empty lower segments of $\boldsymbol{A}$ forms a lattice.

4. [Burris-Sanka. 1.2.5 and 1.3.2] If $\boldsymbol{A} = \langle A, \vee, \wedge \rangle$ is a lattice, then an *ideal* of $\boldsymbol{A}$ is a nonempty lower segment that is closed under $\vee$. Show that the set $I(\boldsymbol{A})$ of ideals of $\boldsymbol{A}$ forms a lattice under $\subseteq$.

   If $\boldsymbol{A}$ is distributive, show that $\langle I(\boldsymbol{A}), \subseteq \rangle$ is distributive.

5. Let $\boldsymbol{A}$ be a bounded lattice (a lattice is *bounded* if it has a least element 0 and a greatest element 1). Let $\mathrm{Sub}(\boldsymbol{A})$ be the set of all sublattices of $\boldsymbol{A}$ that include 0 and 1. Show that $\mathbf{Sub}(\boldsymbol{A}) = \langle \mathrm{Sub}(\boldsymbol{A}), \subseteq \rangle$ is a complete lattice.

   Show that, if $\boldsymbol{A}$ is distributive, then for all $H, K \in \mathrm{Sub}(\boldsymbol{A})$, $H \vee K$ consists of all elements of $\boldsymbol{A}$ of the form $(h_1 \wedge k_1) \vee \cdots \vee (h_n \wedge k_n)$, with $1 \leq n \in \omega$, $h_1, \ldots, h_n \in H$ and $k_1, \ldots, k_n \in K$.