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Abstract

Although most of the familiar logical systems are known to have an algebraic counterpart, no general and precise notion of an algebraizable logic exists upon which a systematic investigation of the process of algebraization can be based. In the memoir such a notion is proposed and the investigation begun.

A deductive system $\mathcal{S}$ over a language $\mathcal{L}$ is *algebraizable* if there exists a quasivariety K of $\mathcal{L}$-algebras such that the $\mathcal{S}$-consequence relation $\vdash_{\mathcal{S}}$ and the equational consequence relation $\models_{\mathsf{K}}$ over K are interpretable in one another in a certain strong sense; K is called an *equivalent algebraic semantics for* $\mathcal{S}$. If $\mathcal{S}$ is algebraizable, then it has precisely one equivalent algebraic semantics. All the logical systems that were known to have an algebraic representation prove to be algebraizable in this precise sense, and in each case the algebraic counterpart turns out to be the equivalent algebraic semantics (up to definitional equivalence).

The main result of the paper is an intrinsic characterization of algebraizability in terms of the *Leibniz operator* Ω, which associates with each theory T of a given deductive system $\mathcal{S}$ a congruence relation ΩT on the formula algebra. ΩT identifies all formulas that cannot be distinguished from one another, on the basis of T, by any property expressible in the language of $\mathcal{S}$. The characterization theorem states that a deductive system $\mathcal{S}$ is algebraizable if and only if Ω is one-to-one and order-preserving on the lattice of $\mathcal{S}$-theories, and in addition preserves directed unions. Several other characterizations are given.

The results and concepts are illustrated by a large number of examples from modal and intuitionistic logic, relevance logic, and classical predicate logic.

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Key-words: deductive system, consequence relation, formula algebra, equational consequence, quasivariety, lattice of theories, universal Horn theory, modal logic, intuitionistic logic, relevance logic, predicate logic.

Contents

Introduction

Algebraic logic in the modern sense can be said to have begun with Tarski's 1935 paper [43] on the foundations of the calculus of systems. Here are found, clearly discernible for the first time, the characteristic features of the subject we recognize today. In the paper Tarski introduced the *algebra of propositional formulas*. He defined a relation $\equiv$ on the set of formulas by the condition

$$\phi \equiv \psi \Leftrightarrow \vdash \phi \to \psi \text{ and} \vdash \psi \to \phi, \tag{1}$$

and asserted that $\equiv$ forms what we now call a *congruence relation* on the algebra of formulas. He then went on to say in effect that the corresponding quotient algebra was a Boolean algebra, and that the theorems of logic (i.e., the tautologies) coincided exactly with the formulas equivalent to $\top$ (or some fixed but arbitrarily chosen tautology). Conversely, Tarski indicated how a deductive system for classical propositional logic can be constructed from any axiomatization of Boolean algebras. Subsequently a number of different non-classical propositional logics were algebraized in this way, the most important being the intuitionistic logic of Heyting, the multiple-valued logics of Post and Lukasiewicz, and the modal logics **S4** and **S5** of Lewis. Tarski himself directed an extensive research program on the algebraization of the classical first-order predicate logic.

The quotient algebra obtained by factoring the algebra of formulas by the congruence (1) has come to be known as the *Tarski-Lindenbaum algebra* of the logic, and the study of an algebraizable logic can to a large extent be reduced to the study of this algebra. There is for example a correspondence between theorems and algebraic identities that allows the deductive apparatus of each algebraizable logic to be interpreted in the equational theory of its Tarski-Lindenbaum algebra. Many higher order metalogical notions also have natural algebraic interpretations; such an interpretation of the deduction theorem is given in [7]. Consequently, when a logic is algebraizable, the powerful methods of modern algebra can be used in its investigation, and this has had a profound influence on the development of these logics.[1]

[1] This is certainly true of the non-classical propositional logics. Algebraic methods have not had as yet a strong impact on metalogical investigations of predicate logic, except in some specialized areas.

There are a number of important logics to which Tarski's method cannot be directly applied, or when it can does not give the expected result. Modal logics, such as the Lewis systems **S1**, **S2**, and **S3**, which do not have the *rule of necessitation* $\vdash \phi \Rightarrow \vdash \Box\phi$, are examples of this kind. In this case the relation defined in (1) is not a congruence on the algebra of formulas because $\vdash \phi \to \psi$ does not imply $\vdash \Box\phi \to \Box\psi$. Other examples are provided by a family of logics that arise from the consideration of a strict (non-material) form of implication where $\varphi \to \psi$ can be a theorem only in the event φ is actually involved in some concrete way in the deduction of ψ. The best known logics of this kind are the systems **R** and **E** of *relevance* and *entailment* found in Anderson and Belnap [2]. In these logics there exist theorems φ and ψ for which the implication $\varphi \to \psi$ fails to be a theorem. Hence the set of all theorems cannot coincide with an equivalence class of the relation $\equiv$ defined in (1).

The question naturally arises if any of these logics can be algebraized by some method other than the classical one, or if they are in a sense inherently non-algebraizable. In order to answer questions like this the notion of an algebraizable logic must be made precise. The problem of formulating such a notion with sufficient degree of generality does not seem to have been addressed in the literature.[2] We propose one in this paper. We believe that it is a very natural one, and that any logic that fails to meet its criteria can with justification be called inherently non-algebraizable. Such a claim cannot of course be established in any absolute, mathematical sense. But to support it we will present several different characterizations of algebraizability apart from the defining condition. They represent natural but quite different aspects of the algebraization process. The fact they all characterize the same notion can be viewed as strong evidence that it is the proper one.

For our purposes a logic is given by an arbitrary set of axioms and inference rules. Logics specified in this way have been called *deductive* or *logistic systems*. More exactly, for us a logic is specified, not just by its set of theorems, but by its consequence relation $\vdash$ thought of as a binary relation between sets of formulas and individual formulas. We want to be able to consider deductive systems, like those of relevance and entailment, that do not have the deduction theorem; in such systems the consequence relation $\vdash$ cannot be defined in terms of the set of theorems. Deductive systems are defined and their elementary properties reviewed in Chapter 1.

The precise definition of an algebraizable logic is given in Chapter 2, Definition 2.10. We first define the notion of algebraic semantics. Roughly speaking,

[2] Other approaches to a general theory of algebraic logic can be found however; see Andréka, Gergely, and Németi [3] and Andréka, Németi, and Sain [4]. (A summary of this work can be found in Henkin, Monk, and Tarski [15, Part II]). See also Felscher and Schulte Mönting [13] and Rasiowa [36].

a class K of algebras is called an *algebraic semantics* for $\mathcal{S}$ if the consequence relation $\vdash_{\mathcal{S}}$ of $\mathcal{S}$ can be interpreted in the (semantical) equational consequence relation $\models_{\mathsf{K}}$ of K in a natural way; K is called an *equivalent algebraic semantics* for $\mathcal{S}$ if there is an inverse interpretation of $\models_{\mathsf{K}}$ in $\vdash_{\mathcal{S}}$. If $\mathcal{S}$ has an equivalent algebraic semantics, we call it *algebraizable*. $\mathcal{S}$ may have many algebraic semantics, but the main theorem of Chapter 2 asserts that a deductive system can be algebraized in essentially only one way (Theorem 2.15).

In Chapter 3 we study the relationship between the theories of a deductive system and the equational theories of its equivalent algebraic semantics. (A *theory* of $\mathcal{S}$ is any set of formulas that contains all axioms and is closed under the inference rules.) This leads to the first characterization of algebraizability. We consider lattices of theories that have been enriched by operators that correspond in a natural way to the substitution functions on the underlying set of formulas. We prove in Theorem 3.7 that K is the equivalent algebraic semantics for a deductive system $\mathcal{S}$ iff there is an isomorphism between the theory lattice of $\mathcal{S}$ and the equational theory lattice of K that commutes with the substitution operators. The proof of this theorem is based in part on a well known technique of universal algebra that originated with Mal′cev [27].

Two intrinsic characterizations of algebraizability are given in Chapter 4, and they have rather different characters. The first one is the main result of the paper and may be of philosophical interest. Let $\mathcal{S}$ be a fixed logic and T a theory of $\mathcal{S}$. It is natural to think of formulas φ and ψ as being equivalent with respect to T if either one can be replaced by the other as a subformula of an arbitrary formula ϑ without affecting the truth or falsity of ϑ relative to T; here the truth or falsity of ϑ is determined by whether or not ϑ is contained in T. The equivalence relation on formulas defined in this way plays an important part in our work and is denoted by $\mathbf{\Omega}T$. The definition of $\mathbf{\Omega}T$ is closely related to the well known method of defining the equality relation in second-order logic that goes back to Leibniz, and in Chapter 1.4 we show that $\mathbf{\Omega}T$ is in fact the natural first-order analogue of the second-order Leibniz relation. For this reason $\mathbf{\Omega}T$ is called the *elementary Leibniz* (*equivalence*) *relation* associated with T. In Theorem 4.2 we prove that for $\mathcal{S}$ to be algebraizable it is necessary that $\mathbf{\Omega}T$ properly includes $\mathbf{\Omega}\mathcal{S}$ whenever T properly includes $\mathcal{S}$, i.e, that the *Leibniz operator* $\mathbf{\Omega}$ is one-one and order-preserving on the lattice of theories of $\mathcal{S}$. We also prove, in what we consider to be the main result of the paper, that this condition is also sufficient for $\mathcal{S}$ to be algebraizable, provided a certain other natural condition holds.

An algebraizable logic can also be characterized by the existence of a finite system of binary composite connectives (i.e., formulas in two variables) that collectively have many of the properties of the biconditional $\leftrightarrow$ of classical logic. This leads to the second intrinsic characterization of algebraizability given in Theorem 4.7; it is an easy consequence of the first. It is especially useful in practice for proving specific deductive systems are algebraizable.

The two intrinsic characterization theorems are used in Chapter 5 to investigate the algebraizability of a number of different deductive systems. We prove there exists a large class of modal logics, including **S1**, **S2**, and **S3**, that are not algebraizable in our sense (Corollary 5.6); hence in our view these logics are intrinsically non-algebraizable. (These logics are however *protoalgebraic* in the sense of [8], and thus amenable to most of the standard methods of algebraic logic; see Chapter 1.4.1.) We also look at the logics of strict implication. We show that relevance logic **R** is algebraizable (Theorem 5.8), while entailment logic **E** is not (Corollary 5.7). On the other hand, the calculus of pure relevant implication $\mathbf{R}_{\rightarrow}$, as well as the calculus of pure entailment $\mathbf{E}_{\rightarrow}$, fails to be algebraizable (Theorem 5.9). (In contrast the implicational fragments of both classical and intuitionistic propositional calculus are algebraizable.) We also investigate the algebraizability of relevance logic when the so-called mingle axiom is adjoined, B-C-I and B-C-K logics, and classical equivalence logic (i.e., the $\{\leftrightarrow\}$-fragment of classical propositional logic). The algebraization of predicate logic presents special problems; these are discussed in Appendix C.

An arbitrary deductive system S can be viewed in a natural way as an elementary (first-order) theory without equality, in fact, as a certain universal Horn theory ES (Chapter 1.3). Algebraizability can then be formulated as an elementary notion, in particular in terms of the definitional equivalence of ES with the elementary theory of a quasi-variety. This is done in Appendix A.