Chapter 5

Matrix Semantics and Algebraizability

We will use the characterization results of the preceding chapter to establish the algebraizability or non-algebraizability of a number of different deductive systems. In particular we settle all the specific questions concerning the algebraizability of systems of non-normal modal logic and of relevance and entailment logic raised in the Introduction. The algebraizability of predicate logic is discussed in Appendix C. In the first part of the chapter we discuss the connection between algebraic and matrix semantics.

Theorem 4.7 seems to be the most useful tool for showing a particular system is algebraizable. Establishing non-algebraizability presents more problems. By Theorem 4.2 a necessary condition for algebraizability is that the Leibniz operator be one-one and order-preserving on the lattice of theories. The theory lattice is usually too complex for this to be usefully applied in practice, at least directly. Presently we show that this property of the Leibniz operator applies to the theory lattices of arbitrary algebras. When the deductive system under consideration has small, finite matrix models this gives a very useful tool for showing non-algebraizability.

5.1 Matrix Semantics and Algebraic Semantics

Let K be a quasivariety over $\mathcal{L}$, and $\mathbf{A}$ an arbitrary $\mathcal{L}$-algebra. A congruence Θ on $\mathbf{A}$ is called a K-*congruence* if $\mathbf{A}/\Theta \in \mathsf{K}$. The K-congruences can be characterized in terms of closure with respect to the equational consequence relation $\models_{\mathsf{K}}$. A congruence relation Θ on an algebra $\mathbf{A}$ is said to be *closed* under a quasi-identity

$$\bigwedge_{i<n} \xi_i(\bar{p}) \approx \eta_i(\bar{p}) \;\rightarrow\; \varphi(\bar{p}) \approx \psi(\bar{p})$$

if, for all $\bar{a} \in A^n$, $\langle \varphi^{\mathbf{A}}(\bar{a}), \psi^{\mathbf{A}}(\bar{a})\rangle \in \Theta$ whenever $\langle \xi_i^{\mathbf{A}}(\bar{a}), \eta_i^{\mathbf{A}}(\bar{a})\rangle \in \Theta$ for all $i < n$. Θ is a K-congruence iff it is closed under each quasi-identity of K, or, equivalently, under each quasi-identity of some base for K. The K-congruences of the formula algebra $\mathbf{Fm}_{\mathcal{L}}$ coincide with the equational theories of K (when $\varphi \approx \psi$ is identified with $\langle \varphi, \psi \rangle$). The set of K-congruences on an $\mathcal{L}$-algebra forms an algebraic lattice. The following theorem may be viewed as a matrix version of the two characterization theorems 3.7 and 4.2.

Theorem 5.1 *Let $\mathcal{S}$ be a deductive system and* K *a quasivariety.*

(i) *The following are equivalent.*

(i′) *$\mathcal{S}$ is algebraizable with equivalent semantics* K.

(i″) *For every algebra* $\mathbf{A}$ *the Leibniz operator* $\boldsymbol{\Omega}_{\mathbf{A}}$ *is an isomorphism between the lattices of $\mathcal{S}$-filters and* K*-congruences of* $\mathbf{A}$.

(ii) *Assume $\mathcal{S}$ is algebraizable with equivalent quasivariety semantics* K. *Let $\delta(p) \approx \epsilon(p)$ be a set of defining equations for* K. *For each algebra* $\mathbf{A}$ *and congruence Θ of* $\mathbf{A}$ *define*

$$H_{\mathbf{A}}\Theta = \left\{ a \in A : \langle \delta^{\mathbf{A}}(a), \epsilon^{\mathbf{A}}(a) \rangle \in \Theta \right\}.$$

Then $H_{\mathbf{A}}$ restricted to the K*-congruences of* $\mathbf{A}$ *is the inverse of* $\boldsymbol{\Omega}_{\mathbf{A}}$.

By taking the algebra $\mathbf{A}$ of (i″) to be $\mathbf{Fm}$ we see that the implication from (i″) to (i′) is an immediate consequence of Theorem 3.7, together with Lemma 4.6.

To get the implication in the opposite direction we need the following matrix version of 4.1.

Lemma 5.2 *Let $\mathcal{S}$ be an algebraizable deductive system over the language $\mathcal{L}$, and let $\Delta(p, q)$ be a system of equivalence formulas. Then*

$$\boldsymbol{\Omega}_{\mathbf{A}}F = \left\{ \langle a, b \rangle : a \, \Delta^{\mathbf{A}} \, b \in F \right\}$$

for every $\mathcal{L}$-algebra $\mathbf{A}$ *and every $\mathcal{S}$-filter F of* $\mathbf{A}$.

Proof. The proof is the matrix-model analogue of the proof of 4.1. Let $\Theta = \{\langle a, b \rangle : a \, \Delta^{\mathbf{A}} \, b \in F\}$. From the definition of a $\mathcal{S}$-filter and the derived inference rule $p \, \Delta \, q, \ q \, \Delta \, r \vdash_{\mathcal{S}} p \, \Delta \, r$ we get that $a \, \Delta^{\mathbf{A}} \, b, \ b \, \Delta^{\mathbf{A}} \, c \in F$ implies $a \, \Delta^{\mathbf{A}} \, c \in F$ for all $a, b \in A$. Hence Θ is transitive. In a similar way we get that Θ is reflexive, symmetric, and has the substitution property: $\langle a_i, b_i \rangle \in \Theta$ for $i = 0, \dots, n-1$ implies $\langle \varphi^{\mathbf{A}}(\bar{a}), \varphi^{\mathbf{A}}(\bar{b}) \rangle \in \Theta$ for every $\varphi(p_0, \dots, p_{n-1}) \in Fm_{\mathcal{L}}$.

By the rule of detachment $p, p \, \Delta q \vdash_{\mathcal{S}} q$ (Theorem 2.14) we have $a, a \, \Delta^{\mathbf{A}} b \in F$ implies $b \in F$ for all $a, b \in A$. Thus Θ is compatible with F, and hence $\Theta \subseteq \boldsymbol{\Omega}_{\mathbf{A}}F$. And since it is clearly elementarily definable over the matrix $\langle \mathbf{A}, F \rangle$ it must coincide with $\boldsymbol{\Omega}_{\mathbf{A}}F$ by Theorem 1.6. ∎

Proof of 5.1. We have already observed that (i″) implies (i′). The implication in the opposite direction can be obtained as a corollary of Theorems 3.7 and 4.2, but a direct proof is conceptually simpler and not much longer.

Assume (i′) holds. Let $\mathbf{A}$ be any algebra and F a $\mathcal{S}$-filter on $\mathbf{A}$. We show that $\mathbf{\Omega_A} F$ is a K-congruence. Suppose $E \models_{\mathsf{K}} \varphi \approx \psi$ and $\langle \xi_{\mathbf{A}}(\bar{a}),\ \eta_{\mathbf{A}}(\bar{a})\rangle \in \mathbf{\Omega_A} F$ for every $\xi \approx \eta \in E$. Then by the lemma

$$\xi_{\mathbf{A}}(\bar{a})\, \Delta^{\mathbf{A}}\, \eta_{\mathbf{A}}(\bar{a}) \in F \quad \text{for every } \xi \approx \eta \in E. \tag{1}$$

But by hypothesis K is an equivalent algebraic semantics for $\mathcal{S}$. Hence by 2.9(i) $E \models_{\mathsf{K}} \varphi \approx \psi$ is equivalent to $\{\xi \,\Delta\, \eta : \xi \approx \eta\} \vdash_{\mathcal{S}} \varphi \,\Delta\, \psi$. So from (1), and the assumption F is a $\mathcal{S}$-filter, we get $\varphi^{\mathbf{A}}(\bar{a})\, \Delta^{\mathbf{A}}\, \psi^{\mathbf{A}}(\bar{a}) \in F$, i.e, $\langle \varphi^{\mathbf{A}}(\bar{a}),\ \psi^{\mathbf{A}}(\bar{a})\rangle \in \mathbf{\Omega_A} F$. Hence $\mathbf{\Omega_A} F$ is closed under K-consequence, and so is a K-congruence.

Now let Θ be an arbitrary K-congruence of $\mathbf{A}$, and let $H_{\mathbf{A}}\Theta$ be the subset of A defined in part (ii) of the theorem. By the dual of the above argument, with 2.8(i) in place of 2.9(i), we get that $H_{\mathbf{A}}\Theta$ is a $\mathcal{S}$-filter. We show that $\mathbf{\Omega_A} H_{\mathbf{A}}\Theta = \Theta$. For all $a, b \in A$ we have $\langle a, b\rangle \in \mathbf{\Omega_A} H_{\mathbf{A}}\Theta$ iff $\langle \delta^{\mathbf{A}}(a\ \Delta^{\mathbf{A}}\ b),\ \epsilon^{\mathbf{A}}(a\ \Delta^{\mathbf{A}}\ b)\rangle \in \Theta$. But by the equational inference rules 2.8(ii):

$$p \approx q \mathrel{=\!\!|\!\!|\!\!=_{\mathsf{K}}} \delta(p\ \Delta\ q) \approx \epsilon(p\ \Delta\ q)$$

we have $\langle \delta^{\mathbf{A}}(a\ \Delta^{\mathbf{A}}\ b),\ \epsilon^{\mathbf{A}}(a\ \Delta^{\mathbf{A}}\ b)\rangle \in \Theta$ iff $\langle a, b\rangle \in \Theta$. Thus $\mathbf{\Omega_A} H_{\mathbf{A}}\Theta = \Theta$ and $\mathbf{\Omega_A}$ is a mapping of the $\mathcal{S}$-filters of $\mathbf{A}$ onto the set of all K-congruences.

It only remains to show $\mathbf{\Omega_A}$ is one-one and order-preserving. From the derived inference rules 2.9(ii): $p \dashv\vdash_{\mathcal{S}} \delta(p)\Delta\epsilon(p)$ we get $a \in F$ iff $\delta^{\mathbf{A}}(a)\Delta^{\mathbf{A}}\epsilon^{\mathbf{A}}(a) \in F$ iff $\langle \delta^{\mathbf{A}}(a),\ \epsilon^{\mathbf{A}}(a)\rangle \in \mathbf{\Omega_A} F$. So for any $\mathcal{S}$-filters F and G we have $F \subset G$ iff $\mathbf{\Omega_A} F \subset \mathbf{\Omega_A} G$.

We have seen that, under the hypothesis of (ii), $\mathbf{\Omega_A} H_{\mathbf{A}}\Theta = \Theta$ for every K-congruence Θ. The dual result, $H_{\mathbf{A}}\mathbf{\Omega_A} F = F$ for every $\mathcal{S}$-filter F, is similarly established with 2.8(ii) in the role of 2.9(ii). ∎

The function $H_{\mathbf{A}}$ from K-congruences to $\mathcal{S}$-filters is in a natural sense the dual of the Leibniz operator. But observe that, while the Leibniz equivalence relation $\mathbf{\Omega_A} F$ is intrinsically defined in terms of $\mathbf{A}$ and the filter set F, the definition of $H_{\mathbf{A}}\Theta$ is given in terms of a set $\delta(p) \approx \epsilon(p)$ of defining equations for K. This is an inherent difference between the two notions; see section 5.2.4 below, especially the introductory remarks.

Theorem 5.1 gives some insight into the precise connection between equivalent algebraic semantics and matrix semantics. One particularly useful way of describing this connection involves the notion of a reduced matrix. An arbitrary matrix $\mathcal{A} = \langle \mathbf{A}, F\rangle$ is *reduced* if $\mathbf{\Omega_A} F = I_A$, the identity relation on A. The class of all reduced $\mathcal{S}$-matrices is easily seen to form a matrix semantics for $\mathcal{S}$ in the sense of Definition 1.3.

Corollary 5.3 *Assume $\mathcal{S}$ is algebraizable. Let K be the equivalent quasivariety semantics and M the class of all reduced $\mathcal{S}$-matrices. Then K is the class of all algebra reducts of M, i.e.,*

$$\mathsf{K} = \{\, \mathbf{A} : \langle \mathbf{A}, F \rangle \in \mathsf{M} \text{ for some } \mathcal{S}\text{-filter } F \,\}.$$

Proof. Let $\langle \mathbf{A}, F \rangle \in \mathsf{M}$. $\mathbf{\Omega_A} F$ is a K-congruence by 5.1(i). Thus $\mathbf{A}/\mathbf{\Omega_A} F \in \mathsf{K}$. On the other hand, $\mathbf{\Omega_A} F = I_A$ since $\langle \mathbf{A}, F \rangle$ is reduced. Thus $\mathbf{A} \in \mathsf{K}$.

Now let $\mathbf{A} \in \mathsf{K}$. Then I_A is a K-congruence, and hence, by 5.1(i), $I_A = \mathbf{\Omega_A} F$ for a (uniquely defined) $\mathcal{S}$-filter F. Thus $\langle \mathbf{A}, F \rangle \in \mathsf{M}$. ∎

Corollary 5.4 *Assume $\mathcal{S}$ is algebraizable. Then $\mathcal{S}$ has the G-rule, i.e., $\varphi, \psi \vdash_{\mathcal{S}} \varphi \,\Delta\, \psi$ (where Δ is any system of equivalence formulas of $\mathcal{S}$,) iff every reduced $\mathcal{S}$-matrix has exactly one designated element.*

Proof. Let K be the equivalent quasivariety semantics for $\mathcal{S}$, and let $\delta \approx \epsilon$ be a system of defining equations. Assume the G-rule holds and $\langle \mathbf{A}, F \rangle$ is a reduced $\mathcal{S}$-matrix. Then $\mathbf{\Omega_A} F = I_A$ and hence

$$F = H_{\mathbf{A}} \mathbf{\Omega_A} F = \{ a \in A : \delta^{\mathbf{A}}(a) = \epsilon^{\mathbf{A}}(a) \}.$$

Suppose $a, b \in F$. Then by the G-rule $a \,\Delta^{\mathbf{A}}\, b \in F$. Thus $\delta^{\mathbf{A}}(a \,\Delta^{\mathbf{A}}\, b) = \epsilon^{\mathbf{A}}(a \,\Delta^{\mathbf{A}}\, b)$. But $\mathbf{A} \in \mathsf{K}$ by 5.3 and

$$\delta(p \,\Delta\, q) \approx \epsilon(p \,\Delta\, q) \models_{\mathsf{K}} p \approx q.$$

Thus $a = b$. So F contains exactly one element.

Assume now that every $\mathcal{S}$-matrix has exactly one designated element. Consider any $\varphi, \psi \in Fm$ and let $T = Cn_{\mathcal{S}}\{\varphi, \psi\}$. $T/\mathbf{\Omega} T$ is the smallest $\mathcal{S}$-filter on the quotient algebra $\mathbf{Fm}/\mathbf{\Omega} T$. For suppose a smaller one exists. Taking its inverse image under the natural homomorphism we would get a $\mathcal{S}$-theory S such that $S \subset T$ and $\mathbf{\Omega} T$ is compatible with S. But this is impossible since $\mathbf{\Omega} S \subset \mathbf{\Omega} T$ and $\mathbf{\Omega} S$ is the largest congruence compatible with S.

By assumption $T/\mathbf{\Omega} T$ contains exactly one element. Thus $\varphi \approx \psi \in \mathbf{\Omega} T$ and hence $\varphi \,\Delta\, \psi \in T$ (by 3.8 and 4.1). Therefore, $\varphi, \psi \vdash_{\mathcal{S}} \varphi \,\Delta\, \psi$ and $\mathcal{S}$ has the G-rule. ∎

According to 5.4, if the G-rule holds, then the class of reduced $\mathcal{S}$-matrices forms an algebraic semantics for $\mathcal{S}$ in the sense of Czelakowski [11].

5.2 Applications and Examples

A somewhat less general algebraic theory of propositional logics has been developed by Rasiowa [36] and Rasiowa and Sikorski [37]. According to [36] a *standard system of implicative extensional propositional calculus* (SIC for short) is a deductive system $\mathcal{S}$ in our sense satisfying the following additional conditions: (i) The language $\mathcal{L}$ of $\mathcal{S}$ contains only a finite number of connectives of rank 0, 1, or 2, and none of higher rank; (ii) $\mathcal{L}$ contains a special binary connective $\to$ for which the following theorems and derived inference rules hold.

$$\vdash_{\mathcal{S}} p \to p, \tag{2}$$

$$p, p \to q \vdash_{\mathcal{S}} q \tag{3}$$

$$p \to q, q \to r \vdash_{\mathcal{S}} p \to r, \tag{4}$$

$$p \vdash_{\mathcal{S}} q \to p, \tag{5}$$

$$p \to q, q \to p \vdash_{\mathcal{S}} Pp \to Pq \quad \text{for every unary } P \in \mathcal{L}, \tag{6}$$

$$p \to q, q \to p, r \to s, s \to r \vdash_{\mathcal{S}} Qpr \to Qqs \quad \text{for every binary } Q \in \mathcal{L}. \tag{7}$$

Take $\Delta(p,q) = \{p \to q, q \to p\}$. Then (2),(4),(6), and (7) give conditions 4.7(i)–(iv). (3) gives the rule of detachment (4.8(v)), and (5) the G-rule (4.8(vi)). Thus by Corollary 4.8, every SIC $\mathcal{S}$ is algebraizable with equivalence system Δ and defining equations $p \approx p \,\Delta\, p$.

The class of SIC logics contains the classical and intuitionistic propositional calculi **PC** and **IPC**, together with almost all their various fragments and extensions that have been considered in the literature. It also contains the normal modal logics and multiple-valued logics. In the remaining part of the section we consider various logics that do not in general fall into this category.

5.2.1 Modal Logics

Let $\mathcal{L} = \{\vee, \wedge, \supset, \neg, \Box\}$. Various deductive systems have appeared in the literature whose theorems coincide with those of Lewis' original **S5**: Cf. Porte [33].

S5$^{\mathrm{G}}$ ("Gödel style" - this is the standard system)

A1 all tautologies
A2 $\Box(p \supset q) \supset (\Box p \supset \Box q)$
A3 $\Box p \supset p$
A4 $\Diamond p \supset \Box\Diamond p$, where $\Diamond p =_{\mathrm{def}} \neg\Box\neg p$
R1 $p, p \supset q \vdash q$
R2 $p \vdash \Box p$.

S5$^{\mathrm{G}}$ is a normal modal logic, and therefore algebraizable.

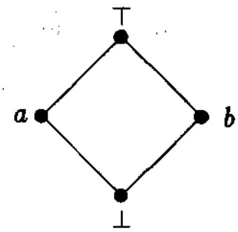

Figure 5.1: **A**

S5$^{\mathrm{C}}$ ("Carnap style")

Axiom A3 and

A1′ $\Box\varphi$, for any tautology φ
A2′ $\Box(\Box(p \supset q) \supset (\Box p \supset \Box q))$
A3′ $\Box(\Box p \supset p)$
A4′ $\Box(\Diamond p \supset \Box\Diamond p)$
R1 $p, p \supset q \vdash q$.

S5$^{\mathrm{W}}$ ("Wajsberg style")

Axioms A1′, A3′, A4′, and

A2″ $\Box(\Box(p \supset q) \supset \Box(\Box p \supset q))$
R3 $p, p \to q \vdash q$, where $p \to q =_{\mathrm{def}} \Box(p \supset q)$.

The compound connective $\to$ is intended to represent the strict implication of Lewis. **S5**$^{\mathrm{W}}$ is closest in spirit to the original **S5**. In particular R3, detachment with respect to strict implication, was also the only rule of inference in Lewis' system. Porte [33] shows that these three systems have the same theorems.

Theorem 5.5 *S5*$^{\mathrm{C}}$ *and* *S5*$^{\mathrm{W}}$ *are not algebraizable.*

Proof. Let **A** be the 4-element modal algebra on $A = \{\bot, a, b, \top\}$, with $\bot < a, b < \top$, $a \not\leq b$, $b \not\leq a$, and $\Box\bot = \Box a = \Box b = \bot$ and $\Box\top = \top$. See Figure 5.1. Let $F_1 = \{a, \top\}$, $F_2 = \{b, \top\}$. F_1 and F_2 are closed under R1 and R3. Furthermore, all axioms of **S5**$^{\mathrm{C}}$ and **S5**$^{\mathrm{W}}$ universally evaluate to $\top$ in **A**. Hence F_1, F_2 are filters with respect to both **S5**$^{\mathrm{C}}$ and **S5**$^{\mathrm{W}}$. It is easy to see that **A** is a simple algebra, i.e., it has no congruence relations other than the identity I_A and the universal relation $A \times A$. Thus $\boldsymbol{\Omega}_{\mathbf{A}} F_1 = \boldsymbol{\Omega}_{\mathbf{A}} F_2 = I_A$. So $\boldsymbol{\Omega}_{\mathbf{A}}$ is not injective on either the **S5**$^{\mathrm{C}}$- or **S5**$^{\mathrm{W}}$-filters of **A**. Consequently, **S5**$^{\mathrm{C}}$ and **S5**$^{\mathrm{W}}$ cannot be algebraized. ∎

The modal system **K** is defined by the axioms A1 and A2 and the inference rules R1 and R2. Let **K**′ be the system whose axioms are the set of theorems of **K** but with R1 (detachment for material implication) as the only inference rule. The axiomatic extensions of **K**′ are called the *quasi-normal* modal systems. $\mathbf{S5}^{\mathrm{W}}$ is quasi-normal. (See Rautenberg [38], Segerberg [39], or Blok and Köhler [6]).

Corollary 5.6 *Every quasi-normal subsystem of* $\mathbf{S5}^{\mathrm{W}}$ *fails to be algebraizable. In particular,* **K**′ *and the Lewis systems* **S1**, **S2**, *and* **S3** *are not algebraizable.* ∎

5.2.2 Entailment and Relevance Logics

In the system **E** of entailment the connectives $\vee$, $\wedge$, and $\neg$ are joined to an entailment connective $\rightarrow$ that, roughly speaking, combines the properties of the strict implication of modal logic with the requirements of relevance. An axiomatization is given in [2, pp.339ff.]. The system **R** of relevance logic is an axiomatic extension of **E**. It is obtained by adding the axiom $p \rightarrow ((p \rightarrow p) \rightarrow p)$. **RM** is the axiomatic extension of **R** by the so-called *mingle* axiom: $p \rightarrow (p \rightarrow p)$.

Corollary 5.7 **E** *is not algebraizable.*

Proof. If we define $\varphi \rightarrow \psi$ to be $\Box(\varphi \supset \psi)$, and $\Box\varphi$ and $\varphi \supset \psi$ to be $(\varphi \rightarrow \varphi) \rightarrow \varphi$ and $\neg\varphi \vee \psi$, respectively, then $\mathbf{S5}^{\mathrm{W}}$ turns out to be an axiomatic extension of **E**. All the axioms of **E** are provable in $\mathbf{S5}^{\mathrm{W}}$. **E** has two rules of inference, R3 and the rule of *conjunction introduction*: $p, q \vdash p \wedge q$. The first is a primitive rule of $\mathbf{S5}^{\mathrm{W}}$, and the second is easily seen to be derivable in $\mathbf{S5}^{\mathrm{W}}$. ∎

An alternative proof would be to verify directly that the sets F_1 and F_2 defined in the proof of Theorem 5.5 are actually **E**-filters of **A**, where $\varphi \rightarrow \psi$ is defined in terms of $\Box$ and $\supset$ as above.

Theorem 5.8 **R** *and* **RM** *are both algebraizable with equivalence formulas* $\{p \rightarrow q, q \rightarrow p\}$ *and the single defining equation* $p \wedge (p \rightarrow p) \approx p \rightarrow p$.

Proof. Let $\Delta(p, q) = \{p \rightarrow q,\ q \rightarrow p\}$. It is known that Δ satisfies 4.7(i)–(iv). (See [2, pp.352f.], or the proof of Theorem 5.10 below.) Now let $\delta(p) = p \wedge (p \rightarrow p)$ and $\epsilon(p) = p \rightarrow p$. We only need to check 4.7(v): $p \dashv\vdash_{\mathbf{R}} \delta(p)\, \Delta\, \epsilon(p)$, i.e.,

$$p \dashv\vdash_{\mathbf{R}} p \wedge (p \rightarrow p)\, \Delta\, (p \rightarrow p).$$

We use the axioms and rules of inference for **R** from [2, p.340]:

A1	$p \to p$
A2	$p \wedge q \to p$
A3	$p \wedge q \to q$
A4	$(p \to q) \wedge (p \to r) \to (p \to (q \wedge r))$
A5	$(p \to ((p \to p) \to p)$
R1	$p,\ p \to q \vdash q$
R2	$p, q \vdash p \wedge q.$

We have the following derivation in **R**.

(1)		$\vdash_{\mathbf{R}}$	$p \to ((p \to p) \to p)$	A5
(2)	p	$\vdash_{\mathbf{R}}$	$(p \to p) \to p$	(1), R1
(3)		$\vdash_{\mathbf{R}}$	$(p \to p) \to (p \to p)$	A1
(4)	p	$\vdash_{\mathbf{R}}$	$((p \to p) \to p)) \wedge ((p \to p) \to (p \to p))$	(2), (3), **R2**
	p	$\vdash_{\mathbf{R}}$	$(p \to p) \to p \wedge (p \to p)$	(4), A4, R1.

Thus, since $\vdash_{\mathbf{R}} p \wedge (p \to p) \to (p \to p)$ by A3, we get $p \vdash_{\mathbf{R}} p \wedge (p \to p) \,\Delta\, p$. For the inference in the other direction observe that

$$(p \to p) \to p \wedge (p \to p) \vdash_{\mathbf{R}} p \wedge (p \to p)$$

by A1 (and modus ponens), and $p \wedge (p \to p) \vdash_{\mathbf{R}} p$ by A6. Thus

$$p \wedge (p \to p) \,\Delta\, p \vdash_{\mathbf{R}} p.$$

So **R** is algebraizable. Since **RM** is an extension of **R** it is algebraizable also. ∎

5.2.3 Pure Implicational Logics

The *pure calculus of entailment* $\mathbf{E}_{\to}$ is a deductive system with $\to$ as the only connective and whose theorems coincide with those of the $\{\to\}$-fragment of **E**. It is axiomatized as follows (see [2, p.79]):

I	$p \to p$	*identity*
B	$(p \to q) \to ((q \to r) \to (p \to r))$	*transitivity*
C′	$(p \to ((s \to t) \to q)) \to ((s \to t) \to (p \to q))$	*restricted commutation*
E	$(p \to (p \to q)) \to (p \to q)$	*contraction*
MP	$p,\ p \to q \vdash q$	*modus ponens.*

We also consider a number of other pure implicational logics that have appeared in the literature. They are based on various combinations of the above

Symbol	Definition	Axioms	Name
$\mathbf{E}_{\rightarrow}$		I,B,C′,E	*pure entailment*
$\mathbf{R}_{\rightarrow}$		I,B,C,E	*relevant implication*
$\mathbf{RMO}_{\rightarrow}$		I,B,C,E,M	
$\mathbf{RM}_{\rightarrow}$	$\{\rightarrow\}$-fragment of **RM**		
BCK		B,C,K	*B-C-K logic*
BCI		I,B,C	*B-C-I logic*
$\mathbf{S5}^{\mathbf{W}}_{\rightarrow}$	$\{\rightarrow\}$-fragment of $\mathbf{S5}^{\mathbf{W}}$		
$\mathbf{IPC}_{\rightarrow}$	$\{\rightarrow\}$-fragment of IPC	D,K	*Hilbert logic*

Table 5.1: Pure Implicational Logics

axioms together with the following:

C	$(p \rightarrow (q \rightarrow r)) \rightarrow (q \rightarrow (p \rightarrow r))$	*commutation*
D	$(p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))$	*distributivity*
K	$p \rightarrow (q \rightarrow p)$	
M	$p \rightarrow (p \rightarrow p)$	*mingle*

In Table 5.1 we list for each logic its symbolic designation, its definition (if other than by its axiomatization), its axiomatization (if known), and its commonly used name (if any). The theorems of $\mathbf{R}_{\rightarrow}$ coincide with those of the $\{\rightarrow\}$-fragment of **R**. Modus ponens (MP) is the only rule of inference in all cases. Observe that the mingle axiom is a substitution instance of K, and hence a theorem of B-C-K logic. Substituting in K we get

$$\vdash_{\mathbf{BCK}} p \rightarrow ((p \rightarrow (p \rightarrow p)) \rightarrow p),$$

and then, by commutation and MP, $\vdash_{\mathbf{BCK}} (p \rightarrow (p \rightarrow p)) \rightarrow (p \rightarrow p)$. Detaching $p \rightarrow (p \rightarrow p)$ gives I. Thus **BCK** is an extension of **BCI**.

The logics listed in Table 5.1 are all distinct from one another. B-C-I logic is the smallest, in the sense that all the others are extensions of it. The known extension relationships between them are given in Figure 5.2. The systems $\mathbf{E}_{\rightarrow}$ and $\mathbf{R}_{\rightarrow}$ correspond to the systems of weak implication considered respectively by Ackermann [1] and Church [10]. The B-C-K and B-C-I logics are presented in Prior [34, p.316]; he attributes them both to C. A. Meredith. They arise naturally in connection with combinatory logic. Axiomatizations of $\mathbf{S5}^{\mathbf{W}}_{\rightarrow}$ are given by C. A. Meredith and Prior in [29].

Theorem 5.9 *None of* **BCI**, $\mathbf{E}_{\rightarrow}$, $\mathbf{S5}^{\mathbf{W}}_{\rightarrow}$, *or* $\mathbf{R}_{\rightarrow}$ *is algebraizable.*

Proof. Let $\mathbf{A} = \langle A, \rightarrow \rangle$ be the algebra with domain $A = \{\top, t, f, \bot\}$ and with $\rightarrow$ defined in the table of Figure 5.3. Note that $\langle \{\top, \bot\}, \rightarrow \rangle$ is

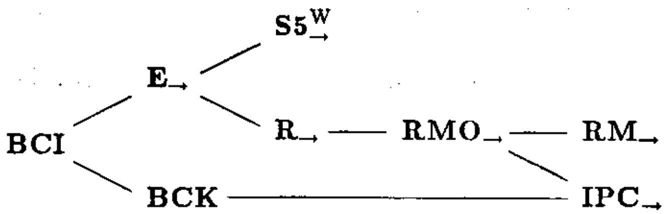

Figure 5.2: Extension relationships

$\rightarrow$	$\top$	t	f	$\bot$
$\top$	$\top$	$\bot$	$\bot$	$\bot$
t	$\top$	t	f	$\bot$
f	$\top$	$\bot$	t	$\bot$
$\bot$	$\top$	$\top$	$\top$	$\top$

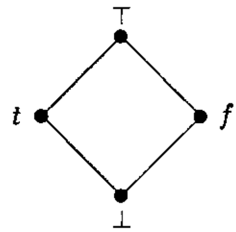

Figure 5.3: **A**

the two-element Hilbert algebra, $\langle\{\top\},\rightarrow\rangle$ and $\langle\{t\},\rightarrow\rangle$ are both one-element subalgebras, and **A** is generated by the element f.

If φ is an $\mathbf{R}_{\rightarrow}$ axiom, then $\varphi^{\mathbf{A}}(\bar{a}) \in \{\top, t\}$ for every interpretation $\bar{a}$ of the variables of φ. This can be verified directly, and also follows from Meyer [30]. Let $F_1 = \{\top, t\}$. Then F_1 is closed under MP and hence is an $\mathbf{R}_{\rightarrow}$-filter. Since $\top \rightarrow \top = \top$ and $\top \rightarrow t = \bot$, $\langle\top,\bot\rangle \in \Theta(\top, t)$ (the congruence on **A** generated by $\langle\top, t\rangle$). Similarly, $\langle\top,\bot\rangle \in \Theta(\bot, f)$. Thus the identity relation I_A is the only congruence compatible with F_1, i.e, $\mathbf{\Omega}_{\mathbf{A}} F_1 = I_A$. Next let $F_2 = \{\top, t, f\}$. F_2 is closed under MP as well, and again $\mathbf{\Omega}_{\mathbf{A}} F_2 = I_A$ since $\langle\top,\bot\rangle \in \Theta(\top, t) \cap \Theta(\top, f) \cap \Theta(t, f)$. Hence $\mathbf{R}_{\rightarrow}$ is not algebraizable by Corollary 5.3. The same applies to **BCI** and $\mathbf{E}_{\rightarrow}$ since $\mathbf{R}_{\rightarrow}$ is an extension of both systems. It does not apply to $\mathbf{S5}^{\mathrm{W}}_{\rightarrow}$ however since this system is not an extension of $\mathbf{R}_{\rightarrow}$. But the proof of non-algebraizability in this case is a straightforward modification of the proof for the full system $\mathbf{S5}^{\mathrm{W}}$.

Let **B** be the $\{\rightarrow\}$-reduct of the 4-element modal algebra defined in the proof of Theorem 5.5 (see Figure 5.1), and let $F_1 = \{\top, a\}$, $F_2 = \{\top, b\}$. We observed in that proof that F_1 and F_2 are $\mathbf{S5}^{\mathrm{W}}$-filters; thus they are *a fortiori* $\mathbf{S5}^{\mathrm{W}}_{\rightarrow}$-filters. But **B**, like **A**, is simple. Because, if $x \not\leq y$, then $\bot \equiv y \rightarrow x \equiv x \rightarrow x \equiv \top$ modulo $\Theta(x, y)$ where $\Theta(x, y)$ is the congruence generated by $\langle x, y\rangle$. So $\mathbf{\Omega}_{\mathbf{B}} F_1 = \mathbf{\Omega}_{\mathbf{B}} F_2$, and $\mathbf{\Omega}_{\mathbf{B}}$ is not one-one on the $\mathbf{S5}^{\mathrm{W}}_{\rightarrow}$-filters of **B**. ∎

The non-algebraizability of $\mathbf{R}_{\rightarrow}$ shows that the role of conjunction in the defining equation for the equivalent quasivariety semantics of **R** is essential.

With regard to the non-algebraizability of **BCI** see Kabziński [19].

Theorem 5.10 $\mathbf{RMO}_{\rightarrow}$, $\mathbf{RM}_{\rightarrow}$, *and* **BCK** *are all algebraizable with equivalence formulas* $\{p \rightarrow q, q \rightarrow p\}$ *and defining equation* $p \approx p \rightarrow p$.

Proof. Let $\mathcal{S}$ be the deductive system defined by the axioms I,B,C,M and MP. $\mathbf{RMO}_{\rightarrow}$, $\mathbf{RM}_{\rightarrow}$, and **BCK** are all extensions of $\mathcal{S}$, so it suffices to prove $\mathcal{S}$ is algebraizable.

Let $\Delta(p,q) = \{p \rightarrow q,\ q \rightarrow p\}$. The derived inference rules $\vdash_{\mathcal{S}} p \,\Delta\, p$ and $p \,\Delta\, q,\ q \,\Delta\, r \vdash_{\mathcal{S}} p \,\Delta\, r$ are immediate consequences of I and B, respectively, and $p \,\Delta\, q \vdash_{\mathcal{S}} q \,\Delta\, p$ is trivial. Thus 4.7(i)–(iii) all hold. By substitution in B we get $(p \rightarrow q) \rightarrow ((q \rightarrow r) \rightarrow (p \rightarrow r))$ and $(q \rightarrow p) \rightarrow ((p \rightarrow r) \rightarrow (q \rightarrow r))$. Thus

$$p \,\Delta\, q \vdash_{\mathcal{S}} (p \rightarrow r) \,\Delta\, (q \rightarrow r). \tag{8}$$

Another substitution in B gives $(q \rightarrow r) \rightarrow ((r \rightarrow s) \rightarrow (q \rightarrow s))$. Permuting the premisses (i.e., using C and MP) we have

$$(r \rightarrow s) \rightarrow ((q \rightarrow r) \rightarrow (q \rightarrow s)),$$

and substitution in this theorem gives $(s \rightarrow r) \rightarrow ((q \rightarrow s) \rightarrow (q \rightarrow r))$. Thus

$$r \,\Delta\, s \vdash_{\mathcal{S}} (q \rightarrow r) \,\Delta\, (q \rightarrow s). \tag{9}$$

From (8) and (9) and the transitivity of Δ we get

$$p \,\Delta\, q,\ q \,\Delta\, p \vdash_{\mathcal{S}} (p \rightarrow r) \,\Delta\, (q \rightarrow s).$$

So 4.7(iv) also holds.

Let $\delta(p) = p$ and $\epsilon(p) = p \rightarrow p$. To establish $p \dashv\vdash_{\mathcal{S}} \delta(p) \,\Delta\, \epsilon(p)$ we must verify

$$p \vdash_{\mathcal{S}} p \rightarrow (p \rightarrow p), \tag{10}$$

$$p \vdash_{\mathcal{S}} (p \rightarrow p) \rightarrow p, \tag{11}$$

$$p \rightarrow (p \rightarrow p),\ (p \rightarrow p) \rightarrow p \vdash_{\mathcal{S}} p. \tag{12}$$

The first follows trivially from the mingle axiom. Permuting the two premisses in $(p \rightarrow p) \rightarrow (p \rightarrow p)$ (by C) we get $\vdash_{\mathcal{S}} p \rightarrow ((p \rightarrow p) \rightarrow p)$. Hence (11) holds. Permutation of the premisses in $((p \rightarrow p) \rightarrow p) \rightarrow ((p \rightarrow p) \rightarrow p)$ and detachment of $p \rightarrow p$ gives $\vdash_{\mathcal{S}} ((p \rightarrow p) \rightarrow p) \rightarrow p$. So (12) holds. ∎

Iseki [16] introduced the notion of BCK-algebra with a single binary operation $*$ and constant 0. The algebras were intended to provide an abstract algebraic model for both the set-theoretical difference operation, and the implication connective of B-C-K logic (actually the dual connective). The following set of axioms is dual to the one given in Iseki and Tanaka [17].

$$(p \to q) \to ((q \to r) \to (p \to r)) \approx \top, \tag{13}$$

$$p \to ((p \to q) \to q) \approx \top, \tag{14}$$

$$p \to p \approx \top, \tag{15}$$

$$p \to \top \approx \top, \tag{16}$$

$$p \to q \approx \top \text{ and } q \to p \approx \top \;\Rightarrow\; p \approx q. \tag{17}$$

It is known that the class of BCK algebras forms an algebraic semantics for **BCK**, in fact it is the smallest quasivariety with this property (Kabziński [20]).

Theorem 5.11 *The class of BCK-algebras is definitionally equivalent to the equivalent quasivariety semantics for* **BCK**.

Proof. Let K be the equivalent quasivariety semantics of **BCK**. By Theorem 5.10 it exists and we can take $p \,\Delta\, q = \{p \to q,\ q \to p\}$ for the equivalence formulas, and $\delta(p) = p$, $\epsilon(p) = p \to p$ for the defining equation.

Substituting in axiom K we get $\vdash_{\mathbf{BCK}} p \to ((q \to q) \to p)$, and then permutation of the premisses gives $\vdash_{\mathbf{BCK}} (q \to q) \to (p \to p)$. Thus

$$\vdash_{\mathbf{BCK}} (p \to p) \,\Delta\, (q \to q),$$

and since K is an equivalent semantics, $\models_{\mathsf{K}} p \to p \approx q \to q$. Let $\top$ be a nullary symbol. For $\mathbf{A} \in \mathsf{K}$ let $\mathbf{A}'$ be the expansion of $\mathbf{A}$ by $\top$, with $\top^{\mathbf{A}} = a \to a$ for some $a \in A$. Let $\mathsf{K}' = \{\mathbf{A}' : \mathbf{A} \in \mathsf{K}\}$. We claim that K' is the class of BCK algebras. Observe that $\langle A, \to, \top\rangle$ belongs to K' iff $\langle A, \to\rangle \in \mathsf{K}$ and $\top^{\mathbf{A}} = a \to a$, for all $a \in A$. Making use of this fact we see that the axiom system for K given by Theorem 2.17 gives the following axiom system for K':

$$\begin{gathered}
(p \to q) \to ((r \to q) \to (r \to p)) \approx \top, \\
(p \to (q \to r)) \to (q \to (p \to r)) \approx \top, \\
p \to (q \to p) \approx \top, \\
p \to p \approx \top, \\
p \approx \top \text{ and } p \to q \approx \top \;\Rightarrow\; q \approx \top, \\
p \to q \approx \top \text{ and } q \to p \approx \top \;\Rightarrow\; p \approx q.
\end{gathered} \tag{18}$$

The only thing that remains is to do now is prove that this set of axioms is equivalent to the axioms (13)–(17) for BCK-algebras. The proof that each axiom of either set is derivable from those of the other set is straightforward with one possible exception: the proof that (18) is a consequence of the BCK axioms; for this see [17, Theorem 1, p.4]. ∎

Wroński [49] has shown that the class of BCK-algebras does not form a variety.

5.2.4 Two Logics with the Same Algebraization

$\mathbf{RMO}_{\rightarrow}$ is an algebraizable, axiomatic extension of $\mathbf{R}_{\rightarrow}$. We now consider two algebraizable, non-axiomatic extensions of $\mathbf{R}_{\rightarrow}$ that are interesting because they are examples of distinct deductive systems with the same equivalent quasivariety semantics. This shows that the dual of Theorem 2.15 fails to hold. It also shows that the inverse $H_{\mathbf{A}}$ of the Leibniz operator (see 5.1(ii)) is not an intrinsic notion of the algebra $\mathbf{A}$.

Let $\mathbf{A} = \langle\{\top, t, f, \bot\}, \rightarrow\rangle$ be the algebra defined in Figure 5.3. Take

$$\delta_1(p) = p, \quad \epsilon_1(p) = p \rightarrow p, \tag{19}$$

$$\delta_2(p) = (p \rightarrow (p \rightarrow p)) \rightarrow p, \quad \epsilon_2(p) = (p \rightarrow (p \rightarrow p)) \rightarrow (p \rightarrow p). \tag{20}$$

Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be the two deductive systems over $\mathcal{L} = \{\rightarrow\}$ defined as follows for $i = 1, 2$.

$$\Gamma \vdash_{\mathcal{S}_i} \varphi \iff \{\delta_i(\psi) \approx \epsilon_i(\psi) : \psi \in \Gamma\} \models_{\mathbf{A}} \delta_i(\varphi) \approx \epsilon_i(\varphi). \tag{21}$$

Since $\mathbf{A}$ is finite, $\models_{\mathbf{A}}$ is finitary, and hence $\mathcal{S}_1, \mathcal{S}_2$ are indeed deductive systems.

Theorem 5.12 *$\mathcal{S}_1$ and $\mathcal{S}_2$ are distinct algebraizable deductive systems with the same equivalent quasivariety semantics $\mathsf{K} = \{\mathbf{A}\}^{\mathrm{Q}}$. K has the same system of equivalence formulas $p \,\Delta\, q = \{p \rightarrow q,\ q \rightarrow p\}$ with respect to both $\mathcal{S}_1$ and $\mathcal{S}_2$, but different defining equations, namely,*

$$\delta_1 = p, \quad \epsilon_1 = p \rightarrow p,$$

$$\delta_2 = \varphi \rightarrow p, \quad \epsilon_2 = \varphi \rightarrow (p \rightarrow p),$$

respectively, where $\varphi = p \rightarrow (p \rightarrow p)$. The two defining equations are not equivalent with respect to K.

Proof. We use Definition 2.8 to show K is the equivalent algebraic semantics for both $\mathcal{S}_1$ and $\mathcal{S}_2$. The first condition of 2.8 coincides with the defining condition for $\mathcal{S}_i$, (21). The other condition of 2.8 is

$$p \approx q \ \mathrel{=\!\mid\!\models_{\mathbf{A}}}\ \delta_i(p \,\Delta\, q) \approx \epsilon_i(p \,\Delta\, q).$$

This reduces to showing that

$$\delta_i(p \mathbin{\Delta} p) \approx \epsilon_i(p \mathbin{\Delta} p), \quad i = 1, 2, \tag{22}$$

are both identities of **A**, and that

$$\delta_i(p \mathbin{\Delta} q) \approx \epsilon_i(p \mathbin{\Delta} q) \;\Rightarrow\; p \approx q, \quad i = 1, 2, \tag{23}$$

are both quasi-identities of **A**.

These can be verified by direct computation, but the computation is simplified by using the two $\mathbf{R}_{\to}$-filters $F_1 = \{\top, t\}$ and $F_2 = \{\top, t, f\}$ introduced in the proof of 5.9. From the table of Figure 5.3 we see that $\delta_1(\top) = \top = \top \to \top = \epsilon_1(\top)$, $\delta_1(t) = t = t \to t = \epsilon_1(t)$, $\delta_1(f) \neq t = f \to f = \epsilon_1(f)$, and $\delta_1(\bot) = \bot \neq \top = \bot \to \bot = \epsilon_1(\bot)$. Thus

$$\delta_1(a) = \epsilon_1(a) \;\Leftrightarrow\; a \in F_1. \tag{24}$$

By a similar direct computation it can be shown that

$$\delta_2(a) = \epsilon_2(a) \;\Leftrightarrow\; a \in F_2. \tag{25}$$

That both equations (22) are identities of **A** follows at once from (24) and (25), together with the fact that $a \mathbin{\Delta} a \in F_1 \subseteq F_2$ for all $a \in A = \{\top, t, f, \bot\}$. To see that (23) are both quasi-identities of **A** it suffices to note that if $a \neq b$, then either $a \to b = \bot$ or $b \to a = \bot$, and hence $\delta_i(a \mathbin{\Delta} b) \neq \epsilon_i(a \mathbin{\Delta} b)$ for $i = 1, 2$.

This shows that $\mathcal{S}_1, \mathcal{S}_2$ are both algebraizable, and that the unique equivalent quasivariety semantics for both of them is the quasivariety generated by **A**. Moreover, $p \mathbin{\Delta} q$ is the unique (in the sense of Theorem 2.13) equivalence system for K with respect to both deductive systems, and $\delta_1(p) = \epsilon_1(p)$ and $\delta_2(p) = \epsilon_2(p)$ are the unique defining equations for K with respect to $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively.

Observe that $f \in F_2$, but $\delta_1(f) \to \epsilon_1(f) = f \to (f \to f) = f \to t = \bot \notin F_2$. Thus, by (24) and (25), $\delta_2(f) = \epsilon_2(f)$ while $\delta_2(\delta_1(f) \to \epsilon_1(f)) \neq \epsilon_2(\delta_1(f) \to \epsilon_1(f))$. So

$$\delta_2(p) \approx \epsilon_2(p) \not\models_{\mathbf{A}} \delta_2(\delta_1(p) \to \epsilon_1(p)) \approx \epsilon_2(\delta_1(p) \to \epsilon_1(p)). \tag{26}$$

From this we conclude by the definition (21) of $\mathcal{S}_1$ that $p \nvdash_{\mathcal{S}_2} \delta_1(p) \to \epsilon_1(p)$, but we do have $p \vdash_{\mathcal{S}_1} \delta_1(p) \to \epsilon_1(p)$ since the interdeducibility relation

$$p \dashv\vdash_{\mathcal{S}_1} \delta_1(p) \mathbin{\Delta} \epsilon_1(p) \tag{27}$$

is a consequence of the fact that $\mathcal{S}_1$ has an equivalent algebraic semantics with equivalence formulas Δ and defining equation $\delta_1 \approx \epsilon_1$. So $\mathcal{S}_1$ and $\mathcal{S}_2$ are distinct deductive systems. Finally, observe that from (21) and (27) we get

$$\delta_1(p) \approx \epsilon_1(p) \models_{\mathbf{A}} \delta_1(\delta_1(p) \to \epsilon_1(p)) \approx \epsilon_1(\delta_1(p) \to \epsilon_1(p)).$$

Comparing this with (26) we see that the equations $\delta_1 \approx \epsilon_1$ and $\delta_2 \approx \epsilon_2$ cannot be equivalent relative to $\mathbf{A}$. ∎

$\mathcal{S}_1$ and $\mathcal{S}_2$ can also be characterized as the deductive systems defined by the matrices $\langle \mathbf{A}, F_1 \rangle$ and $\langle \mathbf{A}, F_2 \rangle$ in the sense that $\Gamma \vdash_{\mathcal{S}_i} \varphi$ iff $\varphi^{\mathbf{A}}(\bar{a}) \in F_i$ whenever $\psi^{\mathbf{A}}(\bar{a}) \in F_i$ for all $\psi \in \Gamma$; this follows immediately from the above proof. Since both matrices are models of $\mathbf{R}_{\to}$, the systems $\mathcal{S}_1$ and $\mathcal{S}_2$ are (non-axiomatic) extensions of $\mathbf{R}_{\to}$. Neither is an extension of $\mathbf{RMO}_{\to}$ since the mingle axiom fails to hold in both $\langle \mathbf{A}, F_1 \rangle$ and $\langle \mathbf{A}, F_2 \rangle$.

5.2.5 Intuitionistic Propositional Logic without Implication

Recall that $\mathbf{IPC}^*$ is the $\{\vee, \wedge, \neg, \top, \bot\}$-fragment of the intuitionistic propositional logic $\mathbf{IPC}$. We showed in Chapter 2.1 that the variety of pseudo-complemented lattices is an algebraic semantics for $\mathbf{IPC}^*$ with defining equation $p \approx \top$. Here we show that $\mathbf{IPC}^*$ is not algebraizable by showing that the Leibniz operator is not order-preserving on the lattice of $\mathbf{IPC}^*$-filters of a finite algebra. Thus $\mathbf{IPC}^*$ is not even protoalgebraic. (See Chapter 1.4.1. By Theorem 4.2 every algebraizable logic is protoalgebraic.) This is the first example we have given of a non-algebraizable deductive system that is not protoalgebraic.

Theorem 5.13 $\mathbf{IPC}^*$ *is not algebraizable. Moreover, it is not even protoalgebraic.*

Proof. Let $\mathbf{A} = \langle \{\top, a, b, \bot\}, \vee, \wedge, \neg, \top, \bot \rangle$ be the 4-element chain pseudo-complemented lattice: $\bot < b < a < \top$, $\neg\top = \neg a = \neg b = \bot$, and $\neg\bot = \top$. Let $F_1 = \{\top\}$ and $F_2 = \{\top, a\}$. $\mathbf{A}$ is the reduct of the 4-element chain Heyting algebra, and F_1 and F_2 are filters of the Heyting algebra, i.e., F_1 and F_2 are $\mathbf{IPC}$-filters. Thus they must also be $\mathbf{IPC}^*$-filters of $\mathbf{A}$. It is an easy matter to check that $\mathbf{\Omega}_{\mathbf{A}} F_1$ identifies only the pair a and b, while $\mathbf{\Omega}_{\mathbf{A}} F_2$ identifies only $\top$ and a. Thus, although $F_1 \subseteq F_2$, we have $\mathbf{\Omega}_{\mathbf{A}} F_1 \not\subseteq \mathbf{\Omega}_{\mathbf{A}} F_2$. Consequently, according to Theorem 5.1(i), $\mathbf{IPC}^*$ cannot be algebraizable. ∎

5.2.6 Equivalential Logic

In contrast to $\mathbf{IPC}^*$ we consider, as a final example, deductive systems without implication that are algebraizable. Let $\mathbf{PC}_\leftrightarrow$ be the $\{\leftrightarrow\}$-fragment of the classical propositional calculus, the (*classical*) *equivalential calculus*. $\mathbf{PC}_\leftrightarrow$ is algebraizable by Corollary 2.12, and its equivalent quasivariety semantics is the class of all subalgebras of $\{+\}$-reducts of Boolean algebras (here $+$ denotes Boolean ring addition). This is just the class BG of Boolean groups (i.e., groups in which every element is of order 2). Leśniewski was the first to produce an axiom system for $\mathbf{PC}_\leftrightarrow$; it contained two axioms. Single axioms were subsequently found by a number of authors including Lukasiewicz. As a final application of Theorem 2.17 we show that Lukasiewicz's axiom system does indeed axiomatize $\mathbf{PC}_\leftrightarrow$; see [26].

Theorem 5.14 $\mathbf{PC}_\leftrightarrow$ *is defined by the single axiom*

(i) $(p \leftrightarrow q) \leftrightarrow ((r \leftrightarrow q) \leftrightarrow (p \leftrightarrow r))$,

and the single rule of inference

(ii) $p, p \leftrightarrow q \vdash q$ *(detachment).*

Proof. Let $\mathcal{S}$ be the deductive system defined by (i) and (ii). Lukasiewicz establishes the following theorems of $\mathcal{S}$; see [26, pp.258–9].

$$\vdash_{\mathcal{S}} p \leftrightarrow p, \tag{28}$$

$$\vdash_{\mathcal{S}} (p \leftrightarrow q) \leftrightarrow (q \leftrightarrow p), \tag{29}$$

$$\vdash_{\mathcal{S}} p \leftrightarrow (q \leftrightarrow (q \leftrightarrow p)). \tag{30}$$

Take $\Delta(p,q) = p \leftrightarrow q$. From axiom (i), (28), and (29) it follows at once by detachment that conditions 4.7(i)–(iv) hold, and (30) together with detachment gives the G-rule $\varphi, \psi \vdash_{\mathcal{S}} \varphi \leftrightarrow \psi$ (4.8(vi)). Thus $\mathcal{S}$ is algebraizable by 4.8, and $p \approx p \,\Delta\, p$ is a single defining equation for its equivalent quasivariety semantics K. From (28) and the G-rule we get $\vdash_{\mathcal{S}} (p \leftrightarrow p) \,\Delta\, (q \leftrightarrow q)$. Thus (replacing $\leftrightarrow$ by $+$ in the semantics) we get $\models_{\mathsf{K}} p + p \approx q + q$. So $p + p$ defines a constant in each member of K that we denote by 0. Applying 2.17 we get the following axiom system for K

$$(p+q) + ((r+q) + (p+r)) \approx 0, \tag{31}$$

$$p + p \approx 0, \tag{32}$$

$$p \approx 0 \text{ and } p + q \approx 0 \Rightarrow q \approx 0, \tag{33}$$

$$p + q \approx 0 \text{ and } q + p \approx 0 \Rightarrow p \approx q. \tag{34}$$

We show that K coincides with the class of Boolean groups. Clearly each of the equations (31)–(34) holds identically in every Boolean group. So $\mathsf{BG} \subseteq \mathsf{K}$. To prove the opposite inclusion let $\mathbf{A} = \langle A, + \rangle \in \mathsf{K}$, and let a, b, c be arbitrary elements of A. By (31) we have $(a+a)+((b+a)+(a+b)) = 0$. Since $a+a = 0$ we get $(b+a)+(a+b) = 0$ by the detachment quasi-identity (33). Similarly $(a+b)+(b+a) = 0$, and hence $a+b = b+a$ by (34). So $\mathbf{A}$ is commutative. (In the sequel we use commutativity repeatedly without further comment.) It follows immediately from (32) and (34) that $a = b$ iff $a+b = 0$. Hence, by (31),

$$a + b = (a+c) + (b+c) \qquad \text{for all } c \in A. \tag{35}$$

So $a+c = b+c$ implies $a = b$, and the cancellation law holds in $\mathbf{A}$. Using (35) we get, for all $a, b, c \in A$,

$$a + (b+c) = (a+b) + ((b+c)+b) \tag{36}$$

Using (35) again we get $(b+c)+c = ((b+c)+b)+(c+b)$, and, canceling $b+c$, $c = (b+c)+b$. Substituting this in (36) gives the associativity of $\mathbf{A}$. Finally, $a+0 = a+(b+b) = (b+a)+b = a$. Thus $\mathbf{A}$ is a Boolean group, and $\mathsf{K} = \mathsf{BG}$.

To complete the proof of the theorem we have only to observe that, since $\mathbf{PC}_{\leftrightarrow}$ and $\mathcal{S}$ both have the same equivalent algebraic semantics with the same defining equation $p \approx p \,\Delta\, p$, they must be equal. ∎

Let $\mathbf{IPC}_{\leftrightarrow}$ be the $\leftrightarrow$-fragment of the intuitionistic propositional calculus, the *intuitionistic equivalential calculus*. A finite axiomatization of the theorems of $\mathbf{IPC}_{\leftrightarrow}$ was given by Tax in [46]; it contains one axiom and the rules

$$p, p \leftrightarrow q \vdash_{\mathcal{S}} q, \qquad p \vdash_{\mathcal{S}} q \leftrightarrow (q \leftrightarrow p).$$

An *equivalential algebra* is an algebra $\langle A, \leftrightarrow \rangle$ satisfying the identities

E1 $(x \leftrightarrow x) \leftrightarrow y \approx y$,
E2 $((x \leftrightarrow y) \leftrightarrow z) \leftrightarrow z \approx (x \leftrightarrow y) \leftrightarrow (y \leftrightarrow z)$,
E3 $((x \leftrightarrow y) \leftrightarrow ((x \leftrightarrow z) \leftrightarrow z)) \leftrightarrow ((x \leftrightarrow z) \leftrightarrow z) \approx x \leftrightarrow y$.

This notion was introduced in [22] by Kabziński and Wroński; see [21] for more details. It follows from their work that every equivalential algebra is a subalgebra of a $\leftrightarrow$-reduct of a Heyting algebra; conversely, every subalgebra of the $\leftrightarrow$-reduct of a Heyting algebra is an equivalential algebra. Hence, by Corollary 2.12, $\mathbf{IPC}_{\leftrightarrow}$ is algebraizable with equivalent quasivariety semantics the variety EA of equivalential algebras and defining equation $p \approx p \leftrightarrow p$. We will now show that Tax' system $\mathcal{S}$ does not only have the same theorems as $\mathbf{IPC}_{\leftrightarrow}$, but actually coincides with it. Take again $\Delta(p, q) = p \leftrightarrow q$. Since

$p \vdash_{\mathcal{S}} q \leftrightarrow (q \leftrightarrow p)$, we have $p, q \vdash_{\mathcal{S}} p \leftrightarrow q$ by detachment, that is, $\mathcal{S}$ has the G-rule. Tax [46] shows that conditions (i)–(iv) of Theorem 4.7 hold, and hence, by Corollary 4.8 $\mathcal{S}$ is algebraizable with equivalent quasivariety semantics K and defining equation $p \approx p \leftrightarrow p$. Using the fact that the theorems of $\mathbf{IPC}_{\leftrightarrow}$ are also theorems of $\mathcal{S}$, we see that K satisfies E1–E3, and thus that $\mathsf{K} \subseteq \mathsf{EA}$. Conversely, from the result that any equivalential algebra is a subalgebra of a reduct of a Heyting algebra it follows easily that the class of equivalential algebras satisfies (i)–(iv) of Theorem 2.17 and hence is that $\mathsf{EA} \subseteq \mathsf{K}$. We have thus shown that the deductive systems $\mathcal{S}$ and $\mathbf{IPC}_{\leftrightarrow}$ have the same equivalent quasivariety semantics; since they also have the same defining equations it follows that they are identical.