Chapter 2

Equational Consequence and Algebraic Semantics

Let $\mathcal{L}$ be a propositional language. By an *$\mathcal{L}$-equation*, or simply an *equation*, we mean a formal expression $\varphi \approx \psi$ where $\varphi, \psi \in Fm_{\mathcal{L}}$. We denote the set of all $\mathcal{L}$-equations by $Eq_{\mathcal{L}}$.

Let K be any class of $\mathcal{L}$-algebras. Let $\models_{\mathsf{K}}$ be the relation that holds between a set Γ of equations and a single equation $\varphi \approx \psi$, in symbols $\Gamma \models_{\mathsf{K}} \varphi \approx \psi$, if every interpretation of $\varphi \approx \psi$ in a member of K holds provided each equation in Γ holds under the same interpretation. Thus $\Gamma \models_{\mathsf{K}} \varphi \approx \psi$ iff, for every $\mathbf{A} \in \mathsf{K}$ and every interpretation $\bar{a}$ of the variables of $\Gamma \cup \{\varphi \approx \psi\}$ as elements of A,

$$\xi^{\mathbf{A}}(\bar{a}) = \eta^{\mathbf{A}}(\bar{a}) \quad \text{for every } \xi \approx \eta \in \Gamma \quad \Rightarrow \quad \varphi^{\mathbf{A}}(\bar{a}) = \psi^{\mathbf{A}}(\bar{a}). \tag{1}$$

In this case we say that $\varphi \approx \psi$ is a *K-consequence* of Γ. The relation $\models_{\mathsf{K}}$ is called the (*semantic*) *equational consequence relation* determined by K.

The K-consequence relation satisfies conditions (1)–(3) of Chapter 1 *mutatis mutandis*, and it is always structural in the sense of (5) (see Lemma 2.1 below). But it need not be finitary. Specifically we say that $\models_{\mathsf{K}}$ is *finitary* if $\Gamma \models_{\mathsf{K}} \varphi \approx \psi$ implies $\Gamma' \models_{\mathsf{K}} \varphi \approx \psi$ for some finite $\Gamma' \subseteq \Gamma$. If $\Gamma = \{\xi_0(\bar{p}) \approx \eta_0(\bar{p}), \dots, \xi_{n-1}(\bar{p}) \approx \eta_{n-1}(\bar{p})\}$ (we write $\bar{p}$ for the sequence $p_0, \dots, p_{n-1}$), then $\Gamma \models_{\mathsf{K}} \varphi(\bar{p}) \approx \psi(\bar{p})$ iff K satisfies the quasi-identities

$$\forall \bar{p}\,(\xi_0(\bar{p}) \approx \eta_0(\bar{p}) \wedge \dots \wedge \xi_{n-1}(\bar{p}) \approx \eta_{n-1}(\bar{p}) \rightarrow \varphi(\bar{p}) \approx \psi(\bar{p})). \tag{2}$$

Thus, if $\models_{\mathsf{K}}$ is finitary, then $\models_{\mathsf{K}}$ coincides with $\models_{\mathsf{K}^Q}$ where K^Q is the quasivariety generated by K. Conversely, if K is a quasivariety, then it is easy to show that $\models_{\mathsf{K}}$ is finitary. Thus, for any class K, $\models_{\mathsf{K}}$ is finitary iff $\models_{\mathsf{K}}$ and $\models_{\mathsf{K}^Q}$ coincide.

The relation $\models_{\mathsf{K}}$ is *structural* if $\Gamma \models_{\mathsf{K}} \varphi \approx \psi$ implies $\sigma(\Gamma) \models_{\mathsf{K}} \sigma\varphi \approx \sigma\psi$ for every substitution σ. ($\sigma(\Gamma) = \{\sigma\xi \approx \sigma\eta : \xi \approx \eta \in \Gamma\}$).

Lemma 2.1 *$\models_{\mathsf{K}}$ is structural for every class K of algebras.*

Proof. For any formula ϑ, substitution σ, and assignment $\bar{a}$ of the variables to elements of A we have $(\sigma\vartheta)^{\mathbf{A}}(\bar{a}) = \vartheta^{\mathbf{A}}(\langle(\sigma p)^{\mathbf{A}}(\bar{a}) : p \text{ a variable}\rangle)$. Thus if (1) holds, then so does the implication

$$(\sigma\xi)^{\mathbf{A}}(\bar{a}) = (\sigma\eta)^{\mathbf{A}}(\bar{a}) \quad \text{for every } \xi \approx \eta \in \Gamma \quad \Rightarrow \quad (\sigma\varphi)^{\mathbf{A}}(\bar{a}) = (\sigma\psi)^{\mathbf{A}}(\bar{a}). \blacksquare$$

2.1 Algebraic Semantics

Definition 2.2 *Let $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$ be a deductive system and K a class of algebras. K is called an* **algebraic semantics** *for $\mathcal{S}$ if $\vdash_{\mathcal{S}}$ can be interpreted in $\models_{\mathsf{K}}$ in the following sense: there exists a finite system $\delta_i(p) \approx \epsilon_i(p)$, for $i < n$, of equations with a single variable p such that, for all $\Gamma \cup \{\varphi\} \subseteq Fm$ and each $j < n$,*

(i) $\Gamma \vdash_{\mathcal{S}} \varphi \Leftrightarrow \{\delta_i[\psi/p] \approx \epsilon_i[\psi/p] : i < n, \psi \in \Gamma\} \models_{\mathsf{K}} \delta_j[\varphi/p] \approx \epsilon_j[\varphi/p]$.

The $\delta_i \approx \epsilon_i$, for $i < n$, are called **defining equations** *for $\mathcal{S}$ and K.*

In order to simplify notation we shall use $\delta(p) \approx \epsilon(p)$ as an abbreviation for a system of defining equations $\delta_i(p) \approx \epsilon_i(p)$, $i < n$. Related abbreviations such as $\delta \approx \epsilon \in \Gamma$ for $\{\delta_i \approx \epsilon_i : i < n\} \subseteq \Gamma$, and $\Gamma \models_{\mathsf{K}} \delta(\varphi) \approx \epsilon(\varphi)$ in place of $\Gamma \models_{\mathsf{K}} \delta_i[\varphi/p] \approx \epsilon_i[\varphi/p]$ for all $i < n$, will also be used (when no confusion is likely) without further explanation. For example, using this abbreviation convention condition 2.2(i) can be written in the more concise form

(i′) $\Gamma \vdash_{\mathcal{S}} \varphi \Leftrightarrow \{\delta(\psi) \approx \epsilon(\psi) : \psi \in \Gamma\} \models_{\mathsf{K}} \delta(\varphi) \approx \epsilon(\varphi)$.

Since $\mathcal{S}$ is always assumed to be finitary, we can also assume without loss of further generality that the set Γ of in 2.2(i′) is always finite.

Assume K is an algebraic semantics for $\mathcal{S}$, and let K^{Q} be the quasivariety generated by K. As previously observed the relation of K-consequence on the right hand side of 2.2(i′) holds iff K satisfies the quasi-identity

$$\bigwedge_{\psi \in \Gamma} \delta(\psi) \approx \epsilon(\psi) \to \delta(\varphi) \approx \epsilon(\varphi).$$

Consequently, 2.2(i′) holds for $\models_{\mathsf{K}}$ iff it holds with $\models_{\mathsf{K}^{\mathsf{Q}}}$ in place of $\models_{\mathsf{K}}$. This gives the following corollary of Definition 2.2.

Corollary 2.3 *If K is an algebraic semantics for a deductive system $\mathcal{S}$, then so is the quasivariety K^{Q}.* $\blacksquare$

An algebraic semantics for $\mathcal{S}$ that is a quasivariety is called a ***quasivariety semantics***. If a deductive system has an algebraic semantics, then by the corollary it also has a quasivariety semantics.

The term algebraic semantics has also been used in a very different sense. For example, Czelakowski [11] defines an algebraic semantics to be a class M of matrices such that M is a matrix semantics for $\mathcal{S}$ in the sense of Definition 1.3 and, in addition, each matrix of M has exactly one designated element.

Let K be an algebraic semantics for $\mathcal{S}$ with defining equations $\delta(p) \approx \epsilon(p)$. For each $\mathbf{A} \in \mathsf{K}$ let $F^{\delta\approx\epsilon}_{\mathbf{A}} = \{ a \in A : \delta^{\mathbf{A}}(a) = \epsilon^{\mathbf{A}}(a)\}$. It is easy to see that $\langle \mathbf{A}, F^{\delta\approx\epsilon}_{\mathbf{A}} \rangle$ is then a $\mathcal{S}$-matrix. In fact we have

Theorem 2.4 *Let $\mathcal{S}$ be a deductive system,* K *a quasivariety, and $\delta(p) \approx \epsilon(p)$ a system of single variable equations. The following are equivalent.*

(i) K *is an algebraic semantics of $\mathcal{S}$ with defining equations $\delta(p) \approx \epsilon(p)$.*

(ii) *The class* $\mathsf{M} = \{ \langle \mathbf{A}, F^{\delta\approx\epsilon}_{\mathbf{A}} \rangle : \mathbf{A} \in \mathsf{K}\}$ *is a matrix semantics for $\mathcal{S}$.*

Proof. This follows immediately from the fact that, for $\Gamma \cup \{\phi\}$,

$$\{\delta(\psi) \approx \epsilon(\psi) : \psi \in \Gamma\} \models_{\mathsf{K}} \delta(\varphi) \approx \epsilon(\varphi) \;\Leftrightarrow\; \Gamma \models_{\mathsf{M}} \varphi.$$

This equivalence in turn is a straightforward consequence of the definitions of M and of the consequences relations $\models_{\mathsf{K}}$ and $\models_{\mathsf{M}}$. ∎

Suppose the language $\mathcal{L}$ has a constant symbol $\top$. As a special case of the above result we get that K is an algebraic semantics for $\mathcal{S}$ with the single defining equation $p \approx \top$ iff $\mathsf{M} = \{\langle \mathbf{A}, \{\top^{\mathbf{A}}\}\rangle : \mathbf{A} \in \mathsf{K}\}$ is a matrix semantics for K. In this case M is an algebraic semantics for $\mathcal{S}$ in the sense of Czelakowski [11].

The variety BA of Boolean algebras, or just the two-element Boolean algebra

$$\mathbf{B} = \langle \{\top, \bot\}, \vee, \wedge, \neg, \top, \bot \rangle$$

alone, is an algebraic semantics for the classical propositional calculus **PC** with the single defining equation $p \approx \top$. Indeed, $\langle \mathbf{B}, \{\top\}\rangle$ is a matrix semantics for **PC**. In this case 2.2(i′) becomes

$$\Gamma \vdash_{\mathbf{PC}} \varphi \;\Leftrightarrow\; \{\psi \approx \top : \psi \in \Gamma\} \models_{\mathsf{BA}} \varphi \approx \top. \tag{3}$$

In fact all deductive systems algebraizable by the classical method have an algebraic semantics with the same defining equation $p \approx \top$; this applies in particular to the intuitionistic propositional calculus, the normal modal logics, and the multiple-valued logics.

An algebraic semantics, if it exists, need not be unique, even if we restrict our attention to quasivarieties. For example, let

$$\mathbf{A} = \langle \{\bot, a, \top\}, \vee, \wedge, \neg, \bot, \top \rangle$$

with $x \vee y = a$ if $x \neq \bot$ or $y \neq \bot$, $\bot \vee \bot = \bot$, $x \wedge y = a$ if $x \neq \bot$ and $y \neq \bot$, $\bot \wedge x = x \wedge \bot = \bot$, $\neg\bot = a$, $\neg\top = \neg a = \bot$. The map $f : \{\bot, a, \top\} \to \{\bot, \top\}$ defined by $f(\bot) = \bot$ and $f(a) = f(\top) = \top$ is a homomorphism from $\mathbf{A}$ to $\mathbf{B}$, and therefore the matrices $\mathcal{A} = \langle \mathbf{A}, \{a, \top\}\rangle$ and $\mathcal{B} = \langle \mathbf{A}, \{\top\}\rangle$ are equivalent in the sense that their corresponding consequence relations $\models_{\mathcal{A}}$ and $\models_{\mathcal{B}}$ coincide. Furthermore, if $\delta_1(p) = \bot \vee p$ and $\epsilon_1(p) = \bot \vee \top$, then for all $x \in \{\bot, a, \top\}$ we have $\delta_1^{\mathbf{A}}(x) = \epsilon_1^{\mathbf{A}}(x)$ iff $x \in \{a, \top\}$. It follows now from Theorem 2.4 that $\mathbf{A}^{\mathrm{Q}}$ (i.e., the quasivariety generated by $\mathbf{A}$) is an algebraic semantics for **PC** with defining equations $\delta_1(p) \approx \epsilon_1(p)$. But $\mathbf{A}^{\mathrm{Q}} \neq \mathsf{BA}$; in fact, $\mathbf{B} \notin \mathbf{A}^{\mathrm{Q}}$ since $\mathbf{A}$ satisfies the quasi-identity

$$\bot \vee \top \approx \top \to \bot \approx \top,$$

while $\mathbf{B}$ does not.[1]

Even if we fix the system of defining equations, an algebraic semantics, if it exists, need not be unique. For example, the quasivariety $\mathsf{Q}_{\mathbf{PC}}$ defined by the identities $\forall \bar{p}(\varphi(\bar{p}) \approx \top)$ for each axiom $\varphi(\bar{p})$ of **PC**, together with the single quasi-identity

$$\forall p \forall q((p \approx \top \wedge (\neg p \vee q) \approx \top) \to q \approx \top))$$

corresponding to the rule of *modus ponens*, is an algebraic semantics for **PC** that includes all Boolean algebras but is clearly much larger. In particular, $\mathsf{Q}_{\mathbf{PC}}$ contains a 3-element chain. (More precisely, $\mathsf{Q}_{\mathbf{PC}}$ contains the non-Boolean algebra $\langle \{\bot, a, \top\}, \wedge, \vee, \neg, \top, \bot\rangle$ where $\langle \{\bot, a, \top\}, \wedge, \vee, \top, \bot\rangle$ forms a lattice with $\bot < a < \top$, and $\neg\bot = \top$, $\neg\top = \bot$, and $\neg a = \top$.) $\mathsf{Q}_{\mathbf{PC}}$ is the largest algebraic semantics for **PC** with the defining equation $p \approx \top$ in the sense that it includes all others. This is an instance of a general phenomenon: If a deductive system $\mathcal{S}$ has any algebraic semantics with defining equations $\delta_i \approx \epsilon_i$, $i < n$, then there is a largest one. It is the quasivariety defined by the identities

$$\delta(\varphi) \approx \epsilon(\varphi)$$

for all axioms φ of $\mathcal{S}$, and the quasi-identities

$$\bigwedge_{j<m} \delta(\varphi_j) \approx \epsilon(\varphi_j) \to \delta(\psi) \approx \epsilon(\psi)$$

for each inference rule $\langle\{\varphi_0, \dots, \varphi_{m-1}\}, \psi\rangle$ of $\mathcal{S}$. (Recall that these expressions are abbreviations for the systems of identities and quasi-identities $\delta_i(\varphi) \approx \epsilon_i(\varphi)$, $i < n$, and

$$\bigwedge_{i<n} \bigwedge_{j<m} \delta_i(\varphi_j) \approx \epsilon_i(\varphi_j) \to \delta_k(\psi) \approx \epsilon_k(\psi), \quad k < n).$$

[1] We are indebted to H. Andréka and I. Németi for the basic idea of this example.

If $\mathcal{S}$ has an algebraic semantics, then so does any fragment of $\mathcal{S}$ whose language includes the defining equations for the semantics. We make this more precise. Let $\mathbf{A} = \langle A, \omega^{\mathbf{A}} \rangle_{\omega \in \mathcal{L}}$ be an $\mathcal{L}$-algebra, and let $\mathcal{L}'$ be any sublanguage of $\mathcal{L}$. The $\mathcal{L}'$-algebra $\langle A, \omega^{\mathbf{A}} \rangle_{\omega \in \mathcal{L}'}$ is called the $\mathcal{L}'$-reduct of $\mathbf{A}$. For any class K of algebras, $\mathbf{S}\,\mathsf{K}$ denotes the class of all algebras isomorphic to a subalgebra of K.

Corollary 2.5 *Let* K *be an algebraic semantics for* $\mathcal{S}$ *with defining equations* $\delta \approx \epsilon$*, and let* $\mathcal{L}'$ *be a sublanguage that contains all the primitive connectives occurring in* $\delta \approx \epsilon$*. Then the class* K' *of all* $\mathcal{L}'$*-reducts of members of* K *is an algebraic semantics for the* $\mathcal{L}'$*-fragment* $\mathcal{S}'$ *of* $\mathcal{S}$*. If* K *is a quasivariety, then* $\mathbf{S}\,\mathsf{K}'$ *is a quasivariety semantics for* $\mathcal{S}'$*.*

Proof. Clearly 2.2(i′) continues to hold when $\mathcal{S}$ is replaced by $\mathcal{S}'$, and K by K'. If K is a quasivariety, then $\mathsf{K}'^{\mathsf{Q}} = \mathbf{S}\,\mathsf{K}'$; see Mal´cev [28, p. 216]. ∎

It is not true in general that the property of having an algebraic semantics is preserved on passing from a deductive system to an extension. Let $\mathcal{S}$ be the trivial system over the language $\mathcal{L} = \{Q\}$ with a single binary connective, where $\Gamma \vdash_{\mathcal{S}} \varphi$ iff $\varphi \in \Gamma$. The class of all $\mathcal{L}$-algebras forms an algebraic semantics for $\mathcal{S}$, but there exist extensions of $\mathcal{S}$ that fail to have any algebraic semantics; see the remarks following Theorem 2.7 below.

Any fragment of **PC**, of the intuitionistic propositional logic, or of any modal logic that is based on **PC** has an algebraic semantics, provided that it contains the truth symbol $\top$ (or that $\top$ is definable in it.) In particular, fragments of these systems in which the implication connective is discarded can have an algebraic semantics. We look at one example of this kind that is of some intrinsic interest.

Let **IPC** denote the intuitionistic propositional calculus over the language $\mathcal{L} = \{\wedge, \vee, \neg, \rightarrow, \top, \bot\}$. The variety HA of Heyting algebras (i.e., relatively pseudo-complemented distributive lattices) is an algebraic semantics for **IPC** with defining equation $p \approx \top$. Let $\mathbf{IPC}^*$ be the deductive system obtained by deleting $\rightarrow$ from **IPC**, i.e., the $\{\wedge, \vee, \neg, \top, \bot\}$-fragment of **IPC**.

For the basic facts about pseudo-complemented and relatively pseudo-complemented distributive lattices used in the following theorem see for instance [5].

Theorem 2.6 *The variety* PCDL *of pseudo-complemented distributive lattices is an algebraic semantics for* $\mathbf{IPC}^*$ *with defining equation* $p \approx \top$*.*

Proof. Let HA^* be the class of $\{\wedge, \vee, \neg, \top, \bot\}$-reducts of HA. By 2.5 HA^* is an algebraic semantics for $\mathbf{IPC}^*$ with defining equation $p \approx \top$. Thus it suffices to show that $\mathsf{PCDL} = \mathbf{S}\,\mathsf{HA}^*$. Clearly $\mathbf{S}\,\mathsf{HA}^* \subseteq \mathsf{PCDL}$. We show that $\mathsf{PCDL}_{SI} \subseteq \mathsf{HA}^*$ where PCDL_{SI} is the class of all subdirectly irreducible members of PCDL.

Let $\mathbf{A} \in \mathsf{PCDL}_{SI}$. Lakser [24] has shown that $\mathbf{A}$ is isomorphic to $\mathbf{B} \oplus \top$ for some Boolean algebra $\mathbf{B}$, where $\mathbf{B} \oplus \top$ is obtained from $\mathbf{B}$ by adjoining a new largest element $\top$ and defining $\neg a$ to be $\bot$ if $a = \top$, $\top$ if $a = \bot$, and the value of $\neg a$ in $\mathbf{B}$ if $a \in B \setminus \{\bot, \top\}$. $\mathbf{A}$ is relatively pseudo-complemented; in fact, the pseudo-complement $a \to b$ of a relative to b is easily seen to be $\neg a \vee b$ if $a \not\leq b$, and $\top$ otherwise. Thus $\mathbf{A}' = \langle A, \wedge, \vee, \neg, \to, \top, \bot \rangle \in \mathsf{HA}$, and hence its $\{\wedge, \vee, \neg, \top, \bot\}$-reduct is in HA^*. From $\mathsf{PCDL}_{SI} \subseteq \mathsf{HA}^*$ it follows at once by Birkhoff's theorem ([14, p. 124]) that $\mathsf{PCDL} \subseteq \mathbf{S}\,\mathsf{HA}^*$. ∎

Not every deductive system has an algebraic semantics. The next theorem shows that the consequence relation of any deductive system $\mathcal{S}$ that is interpretable in the equational logic of a class of algebras is forced to exhibit some of the special characteristics of equational consequence.

Theorem 2.7 *Let $\mathcal{S}$ be a deductive system with algebraic semantics* K *and defining equations $\delta_i \approx \epsilon_i$, for $i < n$. Then $p,\ \delta_i(p) \vdash_{\mathcal{S}} \epsilon_i(p)$ for every $i < n$.*

Proof. Clearly for each $i < n$ we have

$$\delta(p) \approx \epsilon(p),\ \delta(\delta_i(p)) \approx \epsilon(\delta_i(p)) \models_{\mathsf{K}} \delta(\epsilon_i(p)) \approx \epsilon(\epsilon_i(p)).$$

(Recall that $\delta(\varphi) \approx \epsilon(\varphi)$ is an abbreviation for the system of equations $\delta_i(\varphi) \approx \epsilon_i(\varphi), \ldots, \delta_{n-1}(\varphi) \approx \epsilon_{n-1}(\varphi)$.) Applying 2.2(i$'$) we get the conclusion of the theorem. ∎

Let $\mathcal{S}$ be a deductive system with a single, unary connective Q. Assume $\mathcal{S}$ has an algebraic semantics K; let $\delta \approx \epsilon$ be a system of defining equations. If $\mathcal{S}$ is non-trivial in the sense that $\not\vdash_{\mathcal{S}} \varphi$ for at least one φ, then δ_i and ϵ_i must be distinct for some i. Thus $\delta_i \approx \epsilon_i$ is of the form $Q^m p \approx Q^n p$ with $m \neq n$. We can in fact assume without loss of generality that $n = m + 1$. Then by 2.7 we must have $p,\ Q^m p \vdash_{\mathcal{S}} Q^{m+1} p$. It is easy to construct non-trivial $\mathcal{S}$ such that this consequence relation fails to hold for all $m \geq 0$.

It is an open question if any interesting deductive systems fail to have an algebraic semantics in the sense of Definition 2.2. But we have established a partial converse of 2.2, which will be presented elsewhere, that suggests this is unlikely. According to this result every modal logic that includes as theorems all classical tautologies (but not necessarily the rule of necessitation) has an algebraic semantics; this includes the systems **S1** and **S2** of Lewis. The systems **R** and **E** of relevance and entailment and their implicative fragments also turn out to have algebraic semantics. None of these logics is algebraizable in the classical sense, as described in the Introduction, or, with the exception of **R**, in the sense of this paper. See Chapter 5.2 below.

2.2 Equivalent Algebraic Semantics

The variety of Boolean algebras is of course the most important of all the algebraic semantics for **PC**. Its most characteristic property in the present context is that the interpretation of $\vdash_{\mathbf{PC}}$ in $\models_{\mathsf{BA}}$ is invertible in a natural sense. In fact as a kind of dual to the equivalence (3) we have

$$\Gamma \models_{\mathsf{BA}} \varphi \approx \psi \ \Leftrightarrow\ \{\xi \leftrightarrow \eta : \xi \approx \eta \in \Gamma\} \vdash_{\mathbf{PC}} \varphi \leftrightarrow \psi \tag{4}$$

where $\varphi \leftrightarrow \psi = (\neg\varphi \vee \psi) \wedge (\neg\psi \vee \varphi)$ is the usual biconditional. Furthermore, the two interpretability conditions (3) and (4) are inverses of one another in the sense that

$$\varphi \approx \psi \ =\!\mid\models_{\mathsf{BA}} (\varphi \leftrightarrow \psi) \approx \top, \tag{5}$$

$$\varphi \dashv\vdash_{\mathbf{PC}} \varphi \leftrightarrow \top \tag{6}$$

for all $\varphi, \psi \in Fm$. (We have introduced here $\Gamma =\!\mid\models_{\mathsf{K}} \Delta$ as an abbreviation for the conjunction of $\Gamma \models_{\mathsf{K}} \Delta$ and $\Delta \models_{\mathsf{K}} \Gamma$, and similarly for $\dashv\vdash_{\mathcal{S}}$.)

Definition 2.8 *Let $\mathcal{S}$ be an deductive system and* K *an algebraic semantics for $\mathcal{S}$ with defining equations $\delta_i \approx \epsilon_i$, for $i < n$, i.e.,*

(i) $\Gamma \vdash_{\mathcal{S}} \varphi \ \Leftrightarrow\ \{\delta(\psi) \approx \epsilon(\psi) : \psi \in \Gamma\} \models_{\mathsf{K}} \delta(\varphi) \approx \epsilon(\varphi)$.

K *is said to be* **equivalent** *to $\mathcal{S}$ if there exists a finite system $\Delta_j(p, q)$, for $j < m$, of composite binary connectives (i.e, formulas with two variables) such that, for every $\varphi \approx \psi \in Eq$,*

(ii) $\varphi \approx \psi =\!\mid\models_{\mathsf{K}} \delta(\varphi \,\Delta\, \psi) \approx \epsilon(\varphi \,\Delta\, \psi)$.

The system Δ_j, $j < m$, of composite binary connectives satisfying (ii) *is called a system of* **equivalence formulas** *for $\mathcal{S}$ and* K.

(Note that we write $\varphi \,\Delta\, \psi$ in place of $\Delta(\varphi, \psi)$. We have also extended our abbreviation conventions for $\delta_i \approx \epsilon_i$, $i < n$, to the Δ_j, $j < m$, in the obvious way. For instance, 2.8(ii) is shorthand for

$$\varphi \approx \psi =\!\mid\models_{\mathsf{K}} \{\delta_i(\varphi \,\Delta_j\, \psi) \approx \epsilon_i(\varphi \,\Delta_j\, \psi) : i < n, j < m\}.)$$

Of the four conditions, 2.2(i′) and (4)–(6), that characterize BA among the algebraic semantics of **PC**, only the first and third are represented in Definition 2.8. The reason for this is that the other two conditions are logical consequences of these two, and vice versa.

Corollary 2.9 *Let* K *be an algebraic semantics for* $\mathcal{S}$ *with defining equations* $\delta \approx \epsilon$. *If* K *is equivalent to* $\mathcal{S}$ *with equivalence formulas* Δ, *then, for all* $\Gamma \subseteq Eq$ *and each* $\varphi \approx \psi \in Eq$,

(i) $\Gamma \models_{\mathsf{K}} \varphi \approx \psi \;\Leftrightarrow\; \{\xi \,\Delta\, \eta : \xi \approx \eta \in \Gamma\} \vdash_{\mathcal{S}} \varphi \,\Delta\, \psi$,

and, for each $\vartheta \in Fm$,

(ii) $\vartheta \dashv\vdash_{\mathcal{S}} \delta(\vartheta) \,\Delta\, \epsilon(\vartheta)$.

Conversely, if there exists a system of formulas Δ *satisfying conditions* (i) *and* (ii), *then* K *is equivalent to* $\mathcal{S}$ *with equivalence formulas* Δ.

Proof. Assume K is equivalent to $\mathcal{S}$ with equivalence formulas Δ. Then

$$\{\xi \Delta \eta : \xi \approx \eta \in \Gamma\} \vdash_{\mathcal{S}} \varphi \,\Delta\, \psi$$
$$\Leftrightarrow \{\delta(\xi \,\Delta\, \eta) \approx \epsilon(\xi \,\Delta\, \eta) : \xi \approx \eta \in \Gamma\} \models_{\mathsf{K}} \delta(\varphi \,\Delta\, \psi) \approx \epsilon(\varphi \,\Delta\, \psi) \quad \text{by 2.8(i)}$$
$$\Leftrightarrow \{\xi \approx \eta : \xi \approx \eta \in \Gamma\} \models_{\mathsf{K}} \varphi \approx \psi \quad \text{by 2.8(ii)}$$
$$\Leftrightarrow \Gamma \models_{\mathsf{K}} \varphi \approx \psi.$$

Thus (i) holds. To verify (ii) consider any $\vartheta \in Fm$.

$$\vartheta \dashv\vdash_{\mathcal{S}} \delta(\vartheta) \,\Delta\, \epsilon(\vartheta)$$
$$\Leftrightarrow \delta(\vartheta) \approx \epsilon(\vartheta) \mathrel{=\!\!|\!\!\models_{\mathsf{K}}} \delta(\delta(\vartheta) \,\Delta\, \epsilon(\vartheta)) \approx \epsilon(\delta(\vartheta) \,\Delta\, \epsilon(\vartheta)) \quad \text{by 2.8(i)}$$
$$\Leftrightarrow \delta(\vartheta) \approx \epsilon(\vartheta) \mathrel{=\!\!|\!\!\models_{\mathsf{K}}} \delta(\vartheta) \approx \epsilon(\vartheta) \quad \text{by 2.8(ii)}.$$

Thus (ii) also holds. The proof that conditions (i) and (ii) jointly imply 2.8(i),(ii), and hence that K is equivalent to Γ, is similar and will be omitted. ∎

Thus, if K is an equivalent algebraic semantics for $\mathcal{S}$, 2.8(i) guarantees that $\vdash_{\mathcal{S}}$ can be interpreted in $\models_{\mathsf{K}}$, 2.9(i) that $\models_{\mathsf{K}}$ can be interpreted in $\vdash_{\mathcal{S}}$, and 2.8(ii), 2.9(ii) guarantee that these interpretations are, essentially, inverse to one another. This has led us to propose the following

Definition 2.10 *A deductive system* $\mathcal{S}$ *is said to be* **algebraizable** *if it has an equivalent algebraic semantics.*

All the deductive systems that have traditionally formed the subject matter of algebraic logic are algebraizable. Besides **PC** this class of systems includes **IPC** and all of the intermediate logics, as well as the fragments of these logics obtained by restricting the set of primitive connectives in various ways. It also includes the normal modal logics, the multiple-valued logics of Post and

Łukasiewicz, and most of the various versions of quantum logic. In Chapter 5 we investigate the algebraizability of a number of different deductive systems including all those mentioned in the Introduction. We also show that **IPC*** fails to be algebraizable (Theorem 5.13), giving an example of a deductive system with an algebraic semantics but no equivalent algebraic semantics. The algebraizability of predicate logic is discussed in Appendix C.

The duality inherent in the relationship between a deductive system $\mathcal{S}$ and its equivalent algebraic semantics can be viewed as a special case of a general notion of duality associated with any formalization of definitional equivalence; cf. the remarks at the end of Appendix A. The duality between the equivalence formulas Δ and the defining equations $\delta \approx \epsilon$ is just one aspect of this phenomenon. Most statements that can be made about the nature of the relationship between $\mathcal{S}$ and K have a dual form that is obtained by, roughly speaking, interchanging Δ and $\delta \approx \epsilon$, $\mathcal{S}$ and K, and $\vdash_{\mathcal{S}}$ and $\models_{\mathsf{K}}$. The proof of a statement of this kind can also be dualized. Conditions 2.9(i),(ii) are the duals in this sense of 2.8(i),(ii), and the argument that 2.9(i),(ii) jointly imply 2.8(i),(ii), that was omitted in the proof of 2.9, is the dual of the argument that was actually given. The duality between a deductive system and its equivalent algebraic semantics is not perfect however. See Chapter 5.2.4.

Condition 2.8(ii) is equivalent to a system of quasi-identities. This observation together with Corollary 2.3 immediately gives

Corollary 2.11 *Let* K *be an algebraic semantics for a deductive system* $\mathcal{S}$. *Then* K *is equivalent to* $\mathcal{S}$ *iff* K^{Q} *is.* ∎

In view of this corollary and 2.3 we could restrict ourselves exclusively to quasivarieties when considering (equivalent) algebraic semantics. For the most part we will do this; there are certain situations however when it is convenient to consider more general semantics.

Corollary 2.12 *If* $\mathcal{S}$ *is algebraizable, then so is any* $\mathcal{L}'$*-fragment of* $\mathcal{S}$, *where* $\mathcal{L}'$ *contains all the primitive connectives that occur in a system of equivalence formulas and defining equations for the equivalent quasivariety semantics* K *of* $\mathcal{S}$. *Moreover, if* K' *is the class of* $\mathcal{L}'$*-reducts of members of* K, *then* $\mathbf{S}\,\mathsf{K}'$ *is the equivalent quasivariety semantics for the* $\mathcal{L}'$*-fragment of* $\mathcal{S}$.

Proof. Clearly 2.9(i),(ii) continue to hold when $\mathcal{S}$ is replaced by its $\mathcal{L}'$-fragment, and K by K'. By a well known result of Mal´cev [27], if K is a quasivariety, then $(\mathsf{K}')^{\mathsf{Q}} = \mathbf{S}\,\mathsf{K}'$. ∎

It also follows easily from Definition 2.8 that any axiomatic extension of an algebraizable deductive system is itself algebraizable. Actually, this applies to all extensions of $\mathcal{S}$; see Chapter 4.2, Corollary 4.9 below.

2.2.1 Uniqueness

We show that the equivalent quasivariety semantics associated with any fixed algebraizable deductive system is uniquely determined. To prove this we first establish some basic properties of its associated equivalence formulas Δ. The properties all derive from the fact that Δ represents within $\mathcal{S}$ the relation of equality in the algebraic models of $\mathcal{S}$ (see 2.9(i)). The first lemma is based on the fact that the equality relation is a congruence, i.e., an equivalence relation that is preserved by the primitive operations.

Lemma 2.13 *Let Δ be a system of equivalence formulas for some equivalent algebraic semantics for $\mathcal{S}$. Then for all $\varphi, \psi, \vartheta \in Fm$ we have*

(i) $\vdash_{\mathcal{S}} \varphi \,\Delta\, \varphi$;

(ii) $\varphi \,\Delta\, \psi \vdash_{\mathcal{S}} \psi \,\Delta\, \varphi$;

(iii) $\varphi \,\Delta\, \psi \,,\, \psi \,\Delta\, \vartheta \vdash_{\mathcal{S}} \varphi \,\Delta\, \vartheta$.

Also for any variable p occurring in ϑ,

(iv) $\varphi \,\Delta\, \psi \vdash_{\mathcal{S}} \vartheta[\varphi/p] \,\Delta\, \vartheta[\psi/p]$.

Proof. By 2.9(i) we have that (iii) holds iff $\varphi \approx \psi \,,\, \psi \approx \vartheta \models_{\mathsf{K}} \varphi \approx \vartheta$. But the latter consequence is trivial. So (iii) holds, and the other parts of the lemma are obtained the same way. ∎

Lemma 2.14 *Let Δ be as in Lemma 2.13. Then for all $\varphi, \psi \in Fm$ we have $\varphi, \varphi \,\Delta\, \psi \vdash_{\mathcal{S}} \psi$.*

Proof. Let $\delta \approx \epsilon$ be the system of defining equations associated with Δ. It follows from the definition of equational consequence that

$$\delta(\varphi) \approx \epsilon(\varphi) \,,\, \varphi \approx \psi \models_{\mathsf{K}} \delta(\psi) \approx \epsilon(\psi).$$

But by 2.8(ii) $\varphi \approx \psi$ is equivalent in K to $\delta(\varphi \,\Delta\, \psi) \approx \epsilon(\varphi \,\Delta\, \psi)$. Making this substitution in the above relation of consequence we get

$$\delta(\varphi) \approx \epsilon(\varphi) \,,\, \delta(\varphi \,\Delta\, \psi) \approx \epsilon(\varphi \,\Delta\, \psi) \models_{\mathsf{K}} \delta(\psi) \approx \epsilon(\psi).$$

Now applying 2.8(i) we get $\varphi, \varphi \,\Delta\, \psi \vdash_{\mathcal{S}} \psi$. ∎

This lemma shows that Δ satisfies a form of the detachment theorem.

Theorem 2.15 *Let $\mathcal{S}$ be an algebraizable deductive system, and let* K *and* K' *be two equivalent algebraic semantics for $\mathcal{S}$. Then* $\mathsf{K}^{\mathsf{Q}} = \mathsf{K}'^{\mathsf{Q}}$, *i.e.,* K *and* K' *generate the same quasivariety.*

Let Δ and $\delta \approx \epsilon$ be equivalence formulas and defining equations for K, *and similarly Δ' and $\delta' \approx \epsilon'$ for* K'. *Then $\Delta \dashv\vdash_{\mathcal{S}} \Delta'$ and $\delta \approx \epsilon \mathrel{=\joinrel|\joinrel|\joinrel=}_{\mathsf{K}} \delta' \approx \epsilon'$.*

Proof. We first prove that $\Delta \dashv\vdash_{\mathcal{S}} \Delta'$. Taking $\varphi\ \Delta'\ p$ for ϑ in 2.13(iv) we get

$$\varphi\ \Delta\ \psi \vdash_{\mathcal{S}} (\varphi\ \Delta'\ \varphi)\ \Delta\ (\varphi\ \Delta'\ \psi)$$

But $\vdash_{\mathcal{S}} \varphi\ \Delta'\ \varphi$. Thus $\varphi\ \Delta\ \psi \vdash_{\mathcal{S}} \varphi\ \Delta'\ \psi$ by detachment, and by symmetry $\varphi\ \Delta'\ \psi \vdash_{\mathcal{S}} \varphi\ \Delta\ \psi$. Hence $\Delta \dashv\vdash_{\mathcal{S}} \Delta'$.

For any $\Gamma \subseteq Eq$ and $\varphi \approx \psi \in Eq$ we have

$$\begin{aligned} &\Gamma \models_{\mathsf{K}} \varphi \approx \psi \\ &\quad \Leftrightarrow \ \{\xi\ \Delta\ \eta : \xi \approx \eta \in \Gamma\} \vdash_{\mathcal{S}} \varphi\ \Delta\ \psi && \text{by 2.9(i)} \\ &\quad \Leftrightarrow \ \{\xi\ \Delta'\ \eta : \xi \approx \eta \in \Gamma\} \vdash_{\mathcal{S}} \varphi\ \Delta'\ \psi && \text{since } \Delta \dashv\vdash_{\mathcal{S}} \Delta' \\ &\quad \Leftrightarrow \ \Gamma \models_{\mathsf{K}'} \varphi \approx \psi. \end{aligned}$$

So $\models_{\mathsf{K}}$ coincides with $\models_{\mathsf{K}'}$. Let $\bigwedge_{i<n} \xi_i \approx \eta_i \to \varphi \approx \psi$ be any quasi-identity satisfied by K. Then $\{\xi_i \approx \eta_i : i < n\} \models_{\mathsf{K}} \varphi \approx \psi$. Hence

$$\{\xi_i \approx \eta_i : i < n\} \models_{\mathsf{K}'} \varphi \approx \psi.$$

Thus the quasi-identity is also satisfied by K'. Conversely, K satisfies every quasi-identity K' does. So $\mathsf{K}^{\mathsf{Q}} = \mathsf{K}'^{\mathsf{Q}}$. Finally, we prove $\delta \approx \epsilon \mathrel{=\joinrel|\joinrel|\joinrel=}_{\mathsf{K}} \delta' \approx \epsilon'$. We have

$$\begin{aligned} &\delta(p) \approx \epsilon(p) \mathrel{=\joinrel|\joinrel|\joinrel=}_{\mathsf{K}} \delta'(p) \approx \epsilon'(p) \\ &\quad \Leftrightarrow \ \delta(p)\ \Delta\ \epsilon(p) \dashv\vdash_{\mathcal{S}} \delta'(p)\ \Delta\ \epsilon'(p) && \text{by 2.9(i)} \\ &\quad \Leftrightarrow \ \delta(p)\ \Delta\ \epsilon(p) \dashv\vdash_{\mathcal{S}} \delta'(p)\ \Delta'\ \epsilon'(p) && \text{since } \Delta \dashv\vdash_{\mathcal{S}} \Delta' \\ &\quad \Leftrightarrow \ p \dashv\vdash_{\mathcal{S}} p; \end{aligned}$$

the last equivalence follows from 2.9(ii) and the corresponding condition with Δ replaced by Δ'. ∎

The dual of this result fails to hold. There are distinct deductive systems with the same equivalent algebraic semantics. See Chapter 5.2.4.

In Chapter 2.1 we gave an example of a quasivariety semantics for **PC** that does not include BA. Hence the equivalent algebraic semantics of a deductive system $\mathcal{S}$ need not be its smallest algebraic semantics. However we do have

Theorem 2.16 *Let $\mathcal{S}$ be an algebraizable deductive system and* K *the unique quasivariety equivalent to $\mathcal{S}$. Let* K' *be any algebraic semantics for $\mathcal{S}$. Then* K *satisfies every identity that* K' *does. Hence* $\mathsf{K} \subseteq \mathsf{K}'^{\mathsf{V}}$ *where* K'^{V} *is the variety generated by* K'.

Proof. Let Δ be the equivalence formulas associated with K, and let $\delta' \approx \epsilon'$ be the defining equations that go with K′. (Note that since K′ is not assumed to be equivalent to $\mathcal{S}$, we cannot conclude from 2.15 that the $\delta' \approx \epsilon'$ are also defining equations for K.) Let $\varphi \approx \psi$ be any identity of K′. Applying 2.2(i′) with K′ in place of K we get

$$\vdash_{\mathcal{S}} \varphi \,\Delta\, \varphi \Leftrightarrow \models_{\mathsf{K}'} \delta'(\varphi \,\Delta\, \varphi) \approx \epsilon'(\varphi \,\Delta\, \varphi),$$

and hence, since $\varphi \,\Delta\, \varphi$ is a $\mathcal{S}$-theorem, $\models_{\mathsf{K}'} \delta'(\varphi \,\Delta\, \varphi) \approx \epsilon'(\varphi \,\Delta\, \varphi)$. Thus $\models_{\mathsf{K}'} \delta'(\varphi \,\Delta\, \psi) \approx \epsilon'(\varphi \,\Delta\, \psi)$. We now apply 2.2($i'$) again, but in the other direction, to get $\vdash_{\mathcal{S}} \varphi \,\Delta\, \psi$. Finally, apply 2.9(i) to get $\models_{\mathsf{K}} \varphi \approx \psi$. ∎

2.2.2 Axiomatization

There is a simple algorithm for converting any axiomatization of $\mathcal{S}$ into a basis for the quasi-identities of its unique equivalent algebraic semantics.

Theorem 2.17 *Let $\mathcal{S}$ be a deductive system given by a set of axioms Ax and a set of inference rules Ir. Assume $\mathcal{S}$ is algebraizable with equivalence formulas Δ and defining equations $\delta \approx \epsilon$. Then the unique equivalent quasivariety semantics for $\mathcal{S}$ is axiomatized by the identities*

(i) $\delta(\varphi) \approx \epsilon(\varphi)$ *for each* $\varphi \in Ax$,

(ii) $\delta(p \,\Delta\, p) \approx \epsilon(p \,\Delta\, p)$,

together with the following quasi-identities

(iii) $\delta(\psi_0) \approx \epsilon(\psi_0) \wedge \ldots \wedge \delta(\psi_{n-1}) \approx \epsilon(\psi_{n-1}) \;\rightarrow\; \delta(\varphi) \approx \epsilon(\varphi)$

for each $\langle\{\psi_0, \ldots, \psi_{n-1}\}, \varphi\rangle \in Ir$,

(iv) $\delta(p \,\Delta\, q) \approx \epsilon(p \,\Delta\, q) \rightarrow p \approx q$.

Proof. Let L be the quasivariety defined by (i)–(iv). We show that L is the equivalent quasivariety semantics for $\mathcal{S}$. The identity (ii) and quasi-identity (iv) together are equivalent to

$$p \approx q \;=\!\!\mid\models_{\mathsf{L}}\; \delta(p \,\Delta\, q) \approx \epsilon(p \,\Delta\, q),$$

which is the second defining condition 2.8(ii) of an equivalent algebraic semantics. The first condition 2.8(i) is

$$\Gamma \vdash_{\mathcal{S}} \varphi \Leftrightarrow \varphi \in \Phi \tag{7}$$

where

$$\Phi = \{\varphi : \{\delta(\psi) \approx \epsilon(\psi) : \psi \in \Gamma\} \models_{\mathsf{L}} \delta(\varphi) \approx \epsilon(\varphi)\} .$$

From (i) we have $Ax \subseteq \Phi$, and by (iii) Φ is closed under the inference rules of $\mathcal{S}$, i.e., Φ is a $\mathcal{S}$-theory. Thus $\Gamma \vdash_{\mathcal{S}} \varphi$ implies $\varphi \in \Phi$ since Φ includes Γ.

To show the converse assume $\varphi \in \Phi$. Let K be the equivalent quasivariety semantics for $\mathcal{S}$, which exists by hypothesis. K satisfies (i)–(iv), so $\mathsf{K} \subseteq \mathsf{L}$, and hence $\{\delta(\psi) \approx \epsilon(\psi) : \psi \in \Gamma\} \models_{\mathsf{K}} \delta(\varphi) \approx \epsilon(\varphi)$. By 2.8(ii) this equational consequence relation is equivalent to $\Gamma \vdash_{\mathcal{S}} \varphi$. Hence $\varphi \in \Phi$ does imply $\Gamma \vdash_{\mathcal{S}} \varphi$. ∎

By Corollary 2.12 any fragment of **PC**, **IPC**, or modal logic that contains $\rightarrow$ is algebraizable. (In the case of modal logic $\rightarrow$ here refers to material rather than strict implication.) $\mathbf{IPC}_{\rightarrow}$, the $\{\rightarrow\}$-fragment of the intuitionistic propositional calculus, is called *Hilbert logic*, or *positive implication logic*. Its unique equivalent quasivariety semantics is definitionally equivalent to the variety HI of *Hilbert algebras* studied by Diego [12]. As an application of Theorem 2.17 we derive a standard axiomatization of HI.

The system $\mathbf{IPC}_{\rightarrow}$ can be axiomatized by

$$p \rightarrow (q \rightarrow p), \tag{8}$$

$$(p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r)), \tag{9}$$

and the rule of *modus ponens*. Let K be the equivalent quasivariety semantics for $\mathbf{IPC}_{\rightarrow}$, $\Delta(p,q) = \{p \rightarrow q, q \rightarrow p\}$, $\delta(p) = p$, and $\epsilon(p) = p \rightarrow p$. The formulas $(p \rightarrow p)\ \Delta\ (q \rightarrow q)$ are provable in $\mathbf{IPC}_{\rightarrow}$, and hence K satisfies the identity $p \rightarrow p = q \rightarrow q$. Let $\top$ be a nullary symbol, and for each $\mathbf{A} \in \mathsf{K}$ let $\mathbf{A}'$ be the expansion of $\mathbf{A}$ by $\top$, with $\top^{\mathbf{A}'} = a \rightarrow a$ for some $a \in A$. By the above remark, there is only one such algebra $\mathbf{A}'$ for each $\mathbf{A} \in \mathsf{K}$. Let $\mathsf{K}' = \{\mathbf{A}' : \mathbf{A} \in \mathsf{K}\}$. We claim that K' is the class of Hilbert algebras. Note that K' satisfies the identity

$$p \rightarrow p \approx \top, \tag{10}$$

and therefore also $\epsilon(\varphi) \approx \top$ for every formula φ. Using this fact, together with the axiomatization of K obtained by applying Theorem 2.17, we see that K' satisfies the following identities and quasi-identities:

$$p \rightarrow (q \rightarrow p) \approx \top, \tag{11}$$

$$(p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r)) \approx \top, \tag{12}$$

$$p \approx \top \text{ and } p \rightarrow q \approx \top \ \Rightarrow\ q \approx \top, \tag{13}$$

$$p \rightarrow q \approx \top \text{ and } q \rightarrow p \approx \top \ \Rightarrow\ p \approx q. \tag{14}$$

Conversely, any algebra $\langle A, \to, \top \rangle$ satisfying $p \to p \approx \top$ and (11)–(14) belongs to K'. Thus (10)–(14) axiomatize K'.

The variety HI can be defined as the class of algebras $\langle A, \to, \top \rangle$ satisfying (11), (12), (14), and the identity

$$p \to \top = \top; \tag{15}$$

this is proved in Rasiowa [36, pp.22f.]. It is also proved there that every Hilbert algebra satisfies (10). Using this fact it is an easy matter to show that every Hilbert algebra also satisfies (13). Thus $\mathsf{HI} = \mathsf{K}'$.