Chapter 1

Deductive Systems and Matrix Semantics

By a propositional language we will understand some set $\mathcal{L}$ of propositional connectives. The $\mathcal{L}$-*formulas* are built in the usual way from the propositional variables $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \ldots$ using the connectives in $\mathcal{L}$. We denote the set of all $\mathcal{L}$-formulas by $Fm_{\mathcal{L}}$. (The subscript is omitted when the language $\mathcal{L}$ is clear from context.) Light faced italic letters $p, q, r, \ldots$, possibly with subscripts, will be used as metavariables ranging over the set of propositional variables. An assignment $\sigma : \{p_0, p_1, p_2, \ldots\} \to Fm_{\mathcal{L}}$ of formulas to variables extends naturally to a map from $Fm_{\mathcal{L}}$ into itself, also denoted by σ, by setting $\sigma(\phi(p_0, \ldots, p_{n-1})) = \phi(p_0/\sigma p_0, \ldots, p_{n-1}/\sigma p_{n-1})$. σ is called a *substitution*. By a (*finitary*) *inference rule* over $\mathcal{L}$ we mean any pair $\langle \Gamma, \varphi \rangle$ where Γ is a finite set of formulas and φ is a single formula. (From this point of view *modus ponens* would take the form $\langle \{p, p \to q\}, q \rangle$.) A formula ψ is *directly derivable* from a set Δ of formulas by the rule $\langle \Gamma, \varphi \rangle$ if there is a substitution σ such that $\sigma\varphi = \psi$ and $\sigma(\Gamma) \subseteq \Delta$; $(\sigma(\Gamma) = \{\sigma\vartheta : \vartheta \in \Gamma\})$. A *deductive system* $\mathcal{S}$ (*over* $\mathcal{L}$) is defined by a (possible infinite) set of inference rules and axioms; it consists of the pair $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$ where $\vdash_{\mathcal{S}}$ is the relation between sets of formulas and individual formulas defined by the following condition: $\Delta \vdash_{\mathcal{S}} \psi$ iff ψ is contained in the smallest set of formulas that includes Δ together with all substitution instances of the axioms of $\mathcal{S}$, and is closed under direct derivability by the inference rules of $\mathcal{S}$. In informal remarks we often refer to a deductive system as a *logical system* or simply a *logic*. The relation $\vdash_{\mathcal{S}}$ is called the *consequence relation* of $\mathcal{S}$. It is easily seen to satisfy the following three conditions for all $\Gamma, \Delta \subseteq Fm$ and $\varphi, \psi \in Fm$:

$$\varphi \in \Gamma \Rightarrow \Gamma \vdash_{\mathcal{S}} \varphi; \tag{1}$$

$$\Gamma \vdash_{\mathcal{S}} \varphi \text{ and } \Gamma \subseteq \Delta \;\Rightarrow\; \Delta \vdash_{\mathcal{S}} \varphi; \tag{2}$$

$$\Gamma \vdash_{\mathcal{S}} \varphi \text{ and } \Delta \vdash_{\mathcal{S}} \psi \text{ for every } \psi \in \Gamma \;\Rightarrow\; \Delta \vdash_{\mathcal{S}} \varphi. \tag{3}$$

[1] Received by the editor April 19, 1987.
[2] Work partially supported by NSF-grant DMS-8703743.
[3] For a comprehensive account of most of the topics of this chapter see Wójcicki [47],[48].

In addition $\vdash_{\mathcal{S}}$ is *finitary* in the sense

$$\Gamma \vdash_{\mathcal{S}} \varphi \;\Rightarrow\; \Gamma' \vdash_{\mathcal{S}} \varphi \text{ for some finite } \Gamma' \subseteq \Gamma, \tag{4}$$

and it is *structural* in the sense

$$\Gamma \vdash_{\mathcal{S}} \varphi \Rightarrow \sigma(\Gamma) \vdash_{\mathcal{S}} \sigma\varphi \tag{5}$$

for every substitution σ. Conversely, Los and Suszko [25] have shown that every relation satisfying conditions (1)–(5) is the consequence relation for some deductive system $\mathcal{S}$. Consequently, in the sequel by a deductive system we will mean a pair $\langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$ where $\vdash_{\mathcal{S}}$ is a function from the powerset of $Fm_{\mathcal{L}}$ into $Fm_{\mathcal{L}}$ that satisfies conditions (2)–(6); defining axioms and rules of inference are not assumed.

Associated with any deductive system are its various extensions, subsystems, and fragments. By an *extension* of a deductive system $\mathcal{S}$ over the language $\mathcal{L}$ we mean any system $\mathcal{S}' = \langle \mathcal{L}, \vdash_{\mathcal{S}'} \rangle$ over the same language such that $\Gamma \vdash_{\mathcal{S}} \varphi \Rightarrow \Gamma \vdash_{\mathcal{S}'} \varphi$ for all $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$; $\mathcal{S}$ is called a *subsystem* of $\mathcal{S}'$ in this case. $\mathcal{S}'$ is an *axiomatic extension* of $\mathcal{S}$ if it is obtained by adjoining new axioms but leaving the rules of inference fixed.

Let $\mathcal{L}'$ be a sublanguage of $\mathcal{L}$, and let $\vdash_{\mathcal{S}'}$ be the restriction of $\vdash_{\mathcal{S}}$ to $\mathcal{L}$ in the sense that $\Gamma \vdash_{\mathcal{S}'} \varphi$ iff $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{S}'}$ and $\Gamma \vdash_{\mathcal{S}} \varphi$. It is easy to see that conditions (1)–(5) continue to hold with $\vdash_{\mathcal{S}'}$ in place of $\vdash_{\mathcal{S}}$, and hence that $\mathcal{S}' = \langle \mathcal{L}', \vdash_{\mathcal{S}'} \rangle$ is a deductive system over $\mathcal{L}'$. $\mathcal{S}'$ is called the *$\mathcal{L}'$-fragment* of $\mathcal{S}$.

1.1 The Lattice of Theories

Let $\mathcal{S}$ be an arbitrary deductive system. A set T of formulas is called a *theory* of $\mathcal{S}$ (a *$\mathcal{S}$-theory* for short) if $T \vdash_{\mathcal{S}} \varphi$ implies $\varphi \in T$ for every $\varphi \in Fm$, or, equivalently, if T contains all substitution instances of the axioms, and is closed under the inference rules. For any $\Gamma \subseteq Fm$ we take

$$Cn_{\mathcal{S}}\Gamma = \{\varphi \in Fm : \Gamma \vdash_{\mathcal{S}} \varphi\}.$$

$Cn_{\mathcal{S}}\Gamma$ is the smallest $\mathcal{S}$-theory including Γ, and Γ is said to *generate* $Cn_{\mathcal{S}}\Gamma$. $Cn_{\mathcal{S}}$, treated as a function on the power set of Fm, is called the *consequence operator* of $\mathcal{S}$ (the *$\mathcal{S}$-consequence operator*). The characteristic properties (1)–(5) of the consequence relation $\vdash_{\mathcal{S}}$ correspond to the following closure properties of $Cn_{\mathcal{S}}$:

$$\Gamma \subseteq Cn_{\mathcal{S}}\Gamma; \tag{6}$$

$$\Gamma \subseteq \Delta \Rightarrow Cn_{\mathcal{S}}\Gamma \subseteq Cn_{\mathcal{S}}\Delta; \tag{7}$$

$$Cn_{\mathcal{S}}Cn_{\mathcal{S}}\Gamma \subseteq Cn_{\mathcal{S}}\Gamma; \tag{8}$$

$$Cn_{\mathcal{S}}\Gamma \subseteq \bigcup_{\text{finite } \Gamma' \subseteq \Gamma} Cn_{\mathcal{S}}\Gamma'; \tag{9}$$

$$\sigma(Cn_{\mathcal{S}}\Gamma) \subseteq Cn_{\mathcal{S}}\sigma(\Gamma) \tag{10}$$

for every substitution σ. Conversely, any function from the power set of formulas into itself satisfying conditions (6)–(10) gives rise in the obvious way to a deductive system $\langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$. Consequently, $\mathcal{S}$ can be alternatively characterized as the pair $\langle \mathcal{L}, Cn_{\mathcal{S}} \rangle$, and this is how deductive systems have traditionally been defined.[4] Deductive systems in the form $\langle \mathcal{L}, Cn_{\mathcal{S}} \rangle$ are called *standard* systems in Wójcicki [48]. We shall consider exclusively standard systems in this paper, and will thus systematically omit the qualifying term.

The set of all $\mathcal{S}$-theories is denoted by $Th\mathcal{S}$. $Th\mathcal{S}$ is closed under arbitrary intersection and hence forms a complete lattice $\mathbf{Th}\mathcal{S} = \langle Th\mathcal{S}, \cap, \vee^{\mathcal{S}} \rangle$. The largest theory is the set Fm of all formulas, and the smallest is the set of $\mathcal{S}$-*theorems* (i.e., the formulas φ such that $\vdash_{\mathcal{S}} \varphi$, where $\vdash_{\mathcal{S}} \varphi$ stands for $\emptyset \vdash_{\mathcal{S}} \varphi$.) $\vee^{\mathcal{S}}$ denotes the (binary) join operation on $\mathbf{Th}\mathcal{S}$. For any two $\mathcal{S}$-theories T, S we have $T \vee^{\mathcal{S}} S = \bigcap\{R \in Th\mathcal{S} : T \cup S \subseteq R\}$. Alternatively, $T \vee^{\mathcal{S}} S = Cn_{\mathcal{S}}(T \cup S)$; so $T \vee_{\mathcal{S}} S$ is the theory generated by $T \cup S$. The infinitary join operation of $\mathbf{Th}\mathcal{S}$ is denoted by $\bigvee^{\mathcal{S}}$; as in the case of two theories, the join of an arbitrary set $\{T_i : i \in I\}$ of theories can be characterized as the theory generated by $\bigcup_{i \in I} T_i$. Conversely, the consequence operator $Cn_{\mathcal{S}}$, and hence also the consequence relation $\vdash_{\mathcal{S}}$, can be defined in terms of the lattice $\mathbf{Th}\mathcal{S}$. So we could also have characterized a deductive system as the pair $\langle \mathcal{L}, \mathbf{Th}\mathcal{S} \rangle$.

The finitary character of the consequence relation $Cn_{\mathcal{S}}$ has important consequences for the structure of the theory lattice. An element of an arbitrary complete lattice is called *compact* if $a \leq \bigvee X$ implies $a \leq \bigvee X'$ for some finite $X' \subseteq X$. A lattice is *algebraic* if it is complete and every element is the (possibly infinite) join of compact elements. A subset Y of a lattice is *directed* if every finite subset of Y has an upper bound in Y. The proof of the following lemma is straightforward and will be omitted.

Lemma 1.1 *Let $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$ be an arbitrary deductive system.*

(i) *The compact elements of $\mathbf{Th}\mathcal{S}$ coincide with the finitely generated $\mathcal{S}$-theories.*

(ii) *$Th\mathcal{S}$ is closed under unions of directed sets. (I.e., for every directed set $\{T_i : i \in I\}$ of $\mathcal{S}$-theories, $\bigcup_{i \in I} T_i \in Th\mathcal{S}$.)*

(iii) *The lattice $\mathbf{Th}\mathcal{S}$ is algebraic.* ∎

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[4] The notion of deductive system characterized by conditions (6)–(9) (but not (10)) originated with Tarski [42]. He gave it no particular name, but refers to a closely related notion in [43] as a *theory*. In both papers he uses the term *deductive system* to refer to what we call a theory below. Our interchanging of the two terms conforms to current usage in the literature.

Let σ be an arbitrary substitution. $Th\mathcal{S}$ need not be closed under σ, i.e., $\sigma(T)$ may fail to be a theory for some $T \in Th\mathcal{S}$. For this reason we define

$$\sigma_{\mathcal{S}}(T) = Cn_{\mathcal{S}}\sigma(T) \quad \text{for each } T \in Th\mathcal{S}.$$

On the other hand, we shall see in the next lemma that the inverse image of a theory under substitution is always again a theory; this is a consequence of the structurality of $\vdash_{\mathcal{S}}$. For any $\Gamma \subseteq Fm$ let $\sigma^{-1}(\Gamma) = \{\varphi \in Fm : \sigma\varphi \in \Gamma\}$.

Lemma 1.2 *Let $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$ be an arbitrary deductive system.*

(i) *$Th\mathcal{S}$ is closed under inverse substitution. (I.e., $\sigma^{-1}(T) \in Th\mathcal{S}$ for every $T \in Th\mathcal{S}$ and every substitution σ.)*

(ii) *$\sigma_{\mathcal{S}}(Cn_{\mathcal{S}}\Gamma) = Cn_{\mathcal{S}}\sigma(\Gamma)$ for every $\Gamma \subseteq Fm$ and substitution σ.*

(iii) *$\sigma_{\mathcal{S}}$ is a join-continuous mapping of $\mathbf{Th}\mathcal{S}$ into itself. (I.e., we have*

$$\sigma_{\mathcal{S}}(\textstyle\bigvee^{\mathcal{S}}_{i\in I} T_i) = \bigvee^{\mathcal{S}}_{i\in I} \sigma_{\mathcal{S}}(T_i)$$

for any system of theories T_i and any substitution σ.)

Proof. By (10) we have $\sigma(Cn_{\mathcal{S}}\sigma^{-1}(T)) \subseteq Cn_{\mathcal{S}}\sigma(\sigma^{-1}(T)) \subseteq Cn_{\mathcal{S}}T = T$ for every theory T. Thus $Cn_{\mathcal{S}}\sigma^{-1}(T) \subseteq \sigma^{-1}(T)$, and hence (i) holds. (ii) follows easily from (10). To establish (iii) we calculate: $\sigma_{\mathcal{S}}(\bigvee^{\mathcal{S}}_{i\in I} T_i) = \sigma_{\mathcal{S}}(Cn_{\mathcal{S}} \bigcup_{i\in I} T_i) = Cn_{\mathcal{S}}\sigma(\bigcup_{i\in I} T_i) = Cn_{\mathcal{S}} \bigcup_{i\in I} \sigma(T_i) = Cn_{\mathcal{S}} \bigcup_{i\in I} \sigma_{\mathcal{S}}(T_i) = \bigvee^{\mathcal{S}}_{i\in I} \sigma_{\mathcal{S}}(T_i)$. ∎

Observe that condition 2.2 (i) is equivalent to structurality (see Wójcicki [48].)

1.2 Matrix Semantics

By an *$\mathcal{L}$-algebra* we mean a structure $\mathbf{A} = \langle A, \omega^{\mathbf{A}} \rangle_{\omega\in\mathcal{L}}$ where A is a non-empty set, called the *universe* of $\mathbf{A}$, and $\omega^{\mathbf{A}}$ is an operation on A of rank k for each connective ω of rank k. An *$\mathcal{L}$-matrix* is a pair $\mathcal{A} = \langle \mathbf{A}, F \rangle$ where $\mathbf{A}$ is an $\mathcal{L}$-algebra and F is an arbitrary subset of A; the elements of F are called *designated elements* of A. Let M be any class of matrices. Let $\models_{\mathsf{M}}$ be the relation that holds between a (possibly infinite) set Γ of formulas and a single formula φ, in symbols $\Gamma \models_{\mathsf{M}} \varphi$, if every interpretation of φ in a member $\mathcal{A}$ of M holds in $\mathcal{A}$ (i.e., is one of the designated elements) provided each $\psi \in \Gamma$ holds in $\mathcal{A}$ under the same interpretation. For any $\varphi(p_0, \dots, p_{n-1}) \in Fm$ and all $a_0, \dots, a_{n-1} \in A$ we write $\varphi^{\mathbf{A}}(a_0, \dots, a_{n-1})$ for the element of A represented by φ when the variables $p_0, \dots, p_{n-1}$ are interpreted respectively as $a_0, \dots, a_{n-1}$.

Then $\Gamma \models_{\mathsf{M}} \varphi$ iff, for every $\langle \mathbf{A}, F\rangle \in \mathsf{M}$ and every interpretation $\bar{a}$ of the variables of $\Gamma \cup \{\varphi\}$, we have

$$\psi^{\mathbf{A}}(\bar{a}) \in F \text{ for every } \psi \in \Gamma \Rightarrow \varphi^{\mathbf{A}}(\bar{a}) \in F. \tag{11}$$

A matrix $\mathcal{A}$ is called a *matrix model* of $\mathcal{S}$ if $\Gamma \vdash_{\mathcal{S}} \varphi$ implies $\Gamma \models_{\mathcal{A}} \varphi$ for all $\Gamma \cup \{\varphi\} \subseteq Fm$. A subset F of $\mathbf{A}$ is called a $\mathcal{S}$-*filter*, or simply a *filter* when $\mathcal{S}$ is clear from context, if the matrix $\langle \mathbf{A}, F\rangle$ is a matrix model of $\mathcal{S}$. Thus F is a $\mathcal{S}$-filter iff F contains all interpretations of the logical axioms of $\mathcal{S}$ and is closed under each inference rule $\langle \Gamma, \varphi\rangle$ in the sense of (11). The $\mathcal{S}$-filters on the formula algebra are exactly the $\mathcal{S}$-theories defined earlier, and the corresponding matrix models $\langle \mathbf{Fm}, T\rangle$ are called *formula matrix models*.

Definition 1.3 *Let $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}}\rangle$ be a deductive system and* M *a class of $\mathcal{L}$-matrices.* M *is called a* **matrix semantics** *of $\mathcal{S}$ if, for all $\Gamma \cup \{\varphi\} \subseteq Fm$, $\Gamma \vdash_{\mathcal{S}} \varphi \Leftrightarrow \Gamma \models_{\mathsf{M}} \varphi$.*

A class of matrices that forms a matrix semantics for $\mathcal{S}$ in this sense is said to be *strongly adequate* for $\mathcal{S}$. The class of all matrix models forms a matrix semantics for $\mathcal{S}$; so does the class of all formula matrix models.

1.3 Deductive Systems as Elementary Theories

All the various characterizations of a deductive system considered above (in terms of the consequence relation $\vdash_{\mathcal{S}}$, the consequence operator $Cn_{\mathcal{S}}$, or the theory lattice $\mathbf{Th}\mathcal{S}$) are second-order notions. However it can also be characterized in elementary terms as first observed by Bloom [9].

Let $\mathcal{L}$ be the underlying propositional language of $\mathcal{S}$. Let $\mathcal{L}_D$ be the first-order language *without equality* whose extra-logical constants are the primitive connectives of $\mathcal{L}$, now thought of as operation symbols of the appropriate rank, together with a single, unary predicate symbol D. With each axiom φ and each inference rule $\langle\{\psi_0, \ldots, \psi_{n-1}\}, \vartheta\rangle$ of $\mathcal{S}$ we associate respectively the universal positive sentence $\forall \bar{p} D\varphi$ and the universal Horn sentence

$$\forall \bar{p}(D\psi_0 \wedge \ldots \wedge D\psi_{n-1} \rightarrow D\vartheta),$$

where $\bar{p}$ is a list of all variables occurring in $\varphi, \psi_0, \ldots, \psi_{n-1}$, or ϑ. Let ES be the elementary (first-order) universal Horn theory over $\mathcal{L}_D$ axiomatized by these sentences. Then it can be easily shown that, for all $\psi_0, \ldots, \psi_{n-1}, \varphi \in Fm_{\mathcal{L}}$,

$$\psi_0, \ldots, \psi_{n-1} \vdash_{\mathcal{S}} \varphi \Leftrightarrow \vdash_{ES} \forall \bar{p}(D\psi_0 \wedge \ldots \wedge D\psi_{n-1} \rightarrow D\varphi),$$

where $\vdash_{ES}$ denotes consequence relative to ES in the usual first-order sense.

An arbitrary $\mathcal{L}$-matrix $\mathcal{A} = \langle \mathbf{A}, F \rangle$ can be viewed as a structure of the language $\mathcal{L}_D$ where F is the interpretation of the unary predicate symbol D. Then the matrix models of $\mathcal{S}$ coincide exactly with the models of $E\mathcal{S}$ in the usual first-order sense.

1.4 The Elementary Leibniz Equivalence Relation

The identity relation can be defined in second-order logic by the formula

$$p \approx q \leftrightarrow \forall P(P(p) \leftrightarrow P(q))$$

where P is a variable ranging over all unary predicates; this idea goes back to Leibniz. This second-order definition, like the second-order induction axiom, has a first-order analogue in which P is restricted to range over the first-order definable predicates of the underlying structure. We call the relation defined in this way the (*elementary*) *Leibniz* (*equivalence*) *relation*; it plays a fundamental role in the sequel. The Leibniz relation will of course coincide with the identity relation over any language which includes the latter as a primitive; in general this is not the case for languages without equality.

Let $\mathcal{A} = \langle \mathbf{A}, F \rangle$ be any $\mathcal{L}$-matrix. A n-ary predicate P (i.e., relation) is said to be (*elementarily*) *definable* over $\mathcal{A}$ (*with parameters* and *without equality*) if there is a first-order formula $\alpha(p_0, \dots, p_{n-1}, r_0, \dots, r_{k-1})$ in $\mathcal{L}_D$ and elements $c_0, \dots, c_{k-1} \in A$ such that, for all $a_0, \dots, a_{n-1} \in A$,

$$P(a_0, \dots, a_{k-1}) \Leftrightarrow \mathcal{A} \models \alpha[a_0, \dots, a_{n-1}, c_0, \dots, c_{k-1}].$$

The Leibniz relation on a particular matrix will be of less importance to us than the relationship between the various Leibniz relations defined on different matrices with the same underlying algebra.

Definition 1.4 *Let* $\mathbf{A}$ *be any algebra. For each* $F \subseteq A$ *let* $\mathbf{\Omega}_{\mathbf{A}} F$ *be the binary relation on* A *defined by*

$$\mathbf{\Omega}_{\mathbf{A}} F = \{\langle a, b \rangle : P(a) \Leftrightarrow P(b) \text{ for every } P \text{ definable over } \langle \mathbf{A}, F \rangle\}.$$

$\mathbf{\Omega}_{\mathbf{A}} F$ *is called the* **(elementary) Leibniz (equivalence) relation** *on* $\mathbf{A}$ *over* F. *The function* $\mathbf{\Omega}_{\mathbf{A}}$ *with domain the set of all subsets of* A *is called the* **(elementary) Leibniz (equivalence) operator** *on* $\mathbf{A}$.

We shall see below in Theorem 5.1(i) how the behavior of $\mathbf{\Omega}$ on the lattices of $\mathcal{S}$-filters of the matrix-models of $\mathcal{S}$ serves to characterize the algebraizable logics.

We omit the subscript on $\mathbf{\Omega}_{\mathbf{A}}$ when $\mathbf{A}$ is the formula algebra. Thus for any theory T, $\mathbf{\Omega} T = \mathbf{\Omega}_{\mathbf{Fm}} T$.

The equation defining $\boldsymbol{\Omega}_{\mathbf{A}}F$ in 1.4 continues to hold when the range of P is restricted to predicates defined by atomic formulas, i.e., $\mathcal{L}_D$-formulas of the form $D\varphi(p, q_0, \ldots, q_{k-1})$ where φ is an $\mathcal{L}$-formula. If the equivalence $P(a) \Leftrightarrow P(b)$ holds for all predicates of this form, then a simple inductive argument shows that it must also hold for all elementarily definable predicates. Thus we have

$$\boldsymbol{\Omega}_{\mathbf{A}}F = \left\{ \langle a, b\rangle : \begin{array}{l} \varphi^{\mathbf{A}}(a, c_0, \ldots, c_{k-1}) \in F \Leftrightarrow \varphi^{\mathbf{A}}(b, c_0, \ldots, c_{k-1}) \in F \\ \text{for all } \varphi(p, q_0, \ldots, q_{k-1}) \in Fm_{\mathcal{L}} \text{ and all } c_0, \ldots, c_{k-1} \in A \end{array} \right\}.$$

There is still another characterization of the Leibniz relation that is the most useful of all. Recall that an equivalence relation Θ on $\mathbf{A}$ is called a *congruence* if for every $\mathcal{L}$-formula $\varphi(p_0, \ldots, p_{n-1})$ we have

$$\langle \varphi^{\mathbf{A}}(a_0, \ldots, a_{n-1}),\ \varphi^{\mathbf{A}}(b_0, \ldots, b_{n-1})\rangle \in \Theta$$

whenever $\langle a_i, b_i\rangle \in \Theta$ for all $i < n$. A congruence Θ is said to be *compatible* with a subset F of $\mathbf{A}$ if $a \in F$ and $\langle a, b\rangle \in \Theta$ imply $b \in F$ for all $a, b \in A$. (I.e., F is a union of equivalence classes of Θ.)

Theorem 1.5 *For any algebra* $\mathbf{A}$ *and any* $F \subseteq A$, $\boldsymbol{\Omega}_{\mathbf{A}}F$ *is the largest congruence of* $\mathbf{A}$ *compatible with* F.

Proof. $\boldsymbol{\Omega}_{\mathbf{A}}F$ is clearly a congruence and compatible with F. Let Θ be any congruence with this property, and let $\varphi(p,\ q_0, \ldots, q_{k-1})$ be any term of $\mathcal{L}_D$ (i.e., formula of $\mathcal{L}$). Then since Θ is a congruence relation we have, for every $\langle a, b\rangle \in \Theta$ and all $c_0, \ldots, c_{k-1} \in A$,

$$\langle \varphi^{\mathbf{A}}(a,\ c_0, \ldots, c_{k-1}),\ \varphi^{\mathbf{A}}(b,\ c_0, \ldots, c_{k-1})\rangle \in \Theta.$$

Thus by the compatibility of Θ,

$$\varphi^{\mathbf{A}}(a,\ c_0, \ldots, c_{k-1}) \in F \Leftrightarrow \varphi^{\mathbf{A}}(b,\ c_0, \ldots, c_{k-1}) \in F.$$

Hence $\langle a, b\rangle \in \boldsymbol{\Omega}_{\mathbf{A}}F$, and so $\Theta \subseteq \boldsymbol{\Omega}_{\mathbf{A}}F$. ∎

Theorem 1.6 *Let* $\mathbf{A}$ *be an algebra and* $F \subseteq A$. *Let* Θ *be any binary relation on* A *that is elementarily definable over the matrix* $\langle \mathbf{A}, F\rangle$ *with parameters and without equality.*

(i) *If* Θ *is reflexive, then* $\boldsymbol{\Omega}_{\mathbf{A}}F \subseteq \Theta$;

(ii) *if, in addition,* Θ *is a congruence on* $\mathbf{A}$ *that is compatible with* F, *then* $\boldsymbol{\Omega}_{\mathbf{A}}F = \Theta$.

Proof. (i). Let Θ be defined by the formula $\alpha(p, q, r_0, \ldots, r_{k-1})$ of $\mathcal{L}_D$ and the parameters $c_0, \ldots, c_{k-1} \in A$. Let $\langle a, b\rangle \in \boldsymbol{\Omega}_{\mathbf{A}} F$. Since Θ is reflexive, $\langle \mathbf{A}, F\rangle \models \alpha[b, b, c_0, \ldots, c_{k-1}]$. Hence, by the definition of the Leibniz relation, $\langle \mathbf{A}, F\rangle \models \alpha[a, b, c_0, \ldots, c_{k-1}]$, i.e., $\langle a, b\rangle \in \Theta$.

(ii) is an immediate consequence of (i) and Theorem 1.5. ∎

1.4.1 Protoalgebraic Logics

A deductive system $\mathcal{S}$ is called *protoalgebraic* if, for every $\mathcal{S}$-theory T, every pair of $\boldsymbol{\Omega}T$-equivalent formulas are interderivable relative to T, i.e.,

$$\langle \varphi, \psi\rangle \in \boldsymbol{\Omega}T \;\Rightarrow\; T, \varphi \vdash_{\mathcal{S}} \psi \text{ and } T, \psi \vdash_{\mathcal{S}} \varphi.$$

Intuitively this means that, if φ and ψ cannot be distinguished by the deductive apparatus of $\mathcal{S}$ (relative to a fixed but arbitrary $\mathcal{S}$-theory T), then each of φ and ψ is derivable from the other (relative to T). It is not difficult to show that $\mathcal{S}$ is protoalgebraic iff the Leibniz equivalence operator $\boldsymbol{\Omega}$ is order-preserving on the lattice of $\mathcal{S}$-theories in the sense that $T \subseteq S$ implies $\boldsymbol{\Omega}T \subseteq \boldsymbol{\Omega}S$ for all $T, S \in Th\mathcal{S}$.

Protoalgebraic deductive systems are studied in detail in [8]. Although in general not algebraizable in either the classical sense or in the sense of the present paper, they do turn out to be amenable to many of the methods of algebraic logic. Moreover the notion applies to a wide class of logics, in particular all those mentioned in the Introduction.