Fractals and Chaos

A Creative Component

Submitted

In Partial Fulfillment

of the Requirements for the Degree

Master of School Mathematics

Laura Brincks

Iowa State University

November 14, 2005
Introduction – Part 1

After reviewing the last eleven years of Mathematics Teacher magazines, I decided to focus on two closely related topics that I knew little about. They are fractals and chaos. Since, I am also beginning a new position this year, the activities I have chosen for my classroom are not meant to be stand-alone units. Their purpose is to supplement current material and to connect with past lessons.

What are fractals? Benoit Mandelbrot created the term fractal in the 1970s. Fractals are defined as “objects that appear to be broken into many pieces, each piece being a copy of the entire shape” (Naylor, 1999, 360). The name was actually given because fractals exist in fractional dimensions. To visualize a fractal, consider a head of cauliflower or a bunch of broccoli. If a piece is broken off from either of these, the part still resembles the whole. This is the fractal property of self-similarity. In nature, examples of fractals are mountains, coastlines, and rivers.

The fractals closely related cousin is chaos. An illustration of mathematical chaos is Edward Lorenz’s experiment with convection currents (Crownover, 1995). Using the same model, vastly different answers were obtained when Lorenz introduced truncated values halfway through a run. He observed a characteristic of chaos, “sensitive dependence on initial conditions” (130). In layman’s terms, Lorenz referred to it as the “butterfly effect” which questions whether “the flap of a butterfly’s wings in Brazil set off a tornado in Texas” (130).

Fractals have an extensive history in spite of the fact that the word fractal was not introduced until the 1970s. For instance, the Cantor Set (1872), Sierpinski Gasket (1916), and Koch Curve (1904) have been known for quite some time (Peitgen, Jurgens, and Saupe, 1992). However, they were viewed as mathematical monsters. They were used “to demonstrate the deviation from the familiar rather than to typify the normal” (76). For example, the Koch Curve
is familiar to many people because of its snowflake shape. To mathematicians, nevertheless, the Koch Curve was seen as a monster, because its boundary is continuous but not differentiable. As a twenty-year-old student in the 1940s, Mandelbrot recalls being told by a professor “as a topic of active research, the geometry I loved was dead” (9). Furthermore, “it had been dead for nearly a century except in mathematics for children” (10).

In Mandelbrot’s opinion, the turning point in fractal study occurred in 1979-1980 with his research of the Fatou-Julia theory of iteration (Peitgen, Jurgens, and Saupe, 1992). This theory had last been changed in 1918. Mandelbrot used a computer to investigate a small portion of Fatou-Julia, which he referred to as the $\mu$-map. It was later renamed the Mandelbrot Set (M-Set) in his honor by Adrien Douady and John Hubbard. Then, in 1983, Mandelbrot published *The Fractal Geometry of Nature* in which he connected the mathematical monsters of old together into the category of fractal geometry (Camp, 2000).

Mandelbrot credits computers with the awakening of “experimental mathematics” (Peitgen, Jurgens, and Saupe, 1992, 9). The computer not only allowed for enhanced calculations but also provided graphics for previously abstract ideas. As the 1970s progressed, Mandelbrot realized mathematics could be studied using the computer with the same methods he had been applying to physics. It is likely no coincidence that Mandelbrot chose to revisit Fatou-Julia. His uncle, a mathematics professor in France, had provided a twenty year old Mandelbrot with the original reprints of the theory in 1945 in an unsuccessful attempt to try to influence the directions of his nephew’s career. Much to the dismay of the uncle, computers, not pure mathematics, led to the discovery of the Mandelbrot Set. In chaos theory, Lorenz also used computers in the 1961 study of convection currents. He developed a system of twelve ordinary differential equations to simulate the atmosphere. Computers were than used to solve the equations and the data was examined to determine the validity of the model. The chaos finding
ultimately led to Lorenz’s presentation of the “butterfly effect” in 1979 before the American Association of Advancement of Science in which he discussed the impossibility of long-term weather forecasts (Crownover, 1995). The study of fractals and chaos in research and at the collegiate level has been fueled by the advent of the computer. At the secondary mathematics level, graphing calculators lead the charge because of their low cost and ease of portability.

Fractals and chaos are two topics that are very capable of maintaining students’ interests. This is at least partially due to their variety of uses. They are used to study the spread of forest fires and epidemics (Camp, 2000). A fractal process was used to compress images for Microsoft’s Encarta CD-ROM encyclopedia. Richard Voss is currently investigating “how automated fractal interpretation of satellite images can help monitor the health of the environment” (711). Fractals have been used to produce footage for films such as Dante’s Peak and Star Trek II: The Wrath of Khan (Camp, 2000; Crownover, 1995). Fractals are additionally being studied to see if they can be used to identify cancerous tumors (Camp, 2000). Visually, fractals are easier for students to understand than Euclidean geometry. At the same time, fractals and chaos seem to be tailor-made for today’s technologically inclined students.

**Literature Review**

In an article on increasing levels of mathematical maturity, Hare and Philliby give six characteristics of mathematically mature calculus students. Three of the six are relevant here and are listed below.

- **Characteristic 1:** The student demonstrates understanding of important calculus concepts from multiple perspectives.
- **Characteristic 2:** The student has a strong understanding of prerequisite algebra concepts that connect with calculus concepts.
- **Characteristic 3:** The student is able to use technology (2004, 7).
The study of mathematics from multiple perspectives, an understanding of prerequisites, and the use of technology are prevalent throughout the remainder of this section.

A prerequisite for calculus is the understanding of functions. Clement (2001) states, “although the function concept is a central one in mathematics, many research studies of high school students have shown that it is also one of the most difficult for students to understand” (745). Consequently, researchers now advocate approaching functions through multiple representations. These are algebraic, numeric, and graphic (Cunningham 2005; Beckman, Thompson, and Senk, 1999; Keller and Hirsch, 1998). According to Robert Cunningham, there are six types of transfers between representations. They are illustrated below.

Cunningham found that students have difficulty changing between representations to solve problems. Most troubling of all are transfers involving graphs. Six transfer problem types from his study follow.

- **A = Algebraic**
  - **N = Numeric**
  - **G = Graphic**

**A to G**  Graph the linear equation $3x + 2y = 12$ using slope and $y$-intercept.

**G to A**  Given the graph shown, determine the linear equation represented. (Assume each tick mark represents one unit on the graph)

- **A to N**  Given the linear equation $3x + 2y = 12$, construct a table displaying ordered pairs that are solutions to the equation.

- **N to A**  Given a set of two ordered pairs, determine the equation of the line that passes through them.
One such study that Cunningham (2005) reviews involving functions and transfers was completed in 1999 by Porzio. In the study, Porzio compared three sections of calculus. Their approaches included: traditional, graphing calculator, and the use of Mathematica. The traditional and graphing calculator sections performed poorly on transfer problems such as the following exercise from their posttest.

The population of a herd of deer is given by the function \( P(t) = 4000 - 500\cos2\pi t \) where \( t \) is measured in years and \( t = 0 \) corresponds to January 1.

a) When in the year is the population at its maximum? What is that maximum?
b) When in the year is the population at its minimum? What is that minimum?
c) When in the year is the population increasing the fastest?
d) Approximately how fast is the population changing on the first of July?

(Porzio, 1999, 11)

Porzio found that the Mathematica group did much better since “they had been given many opportunities to solve transfer problems” (Cunningham, 2005, 74). The graphing calculator class had seen multiple representations modeled. However, they had not been made to practice transferring on their own. Porzio’s conclusion, which has been duplicated several times since 1999, is that teachers must require students to solve problems and make connections using multiple representations (Cunningham 2005; Beckman, Thompson, and Senk, 1999).

Demonstration alone is not effective in building an understanding of functions. In part three, of this paper, graphing calculators will be utilized in high school mathematics classes as part of a multi-representational study of functions related to fractals.

Another important idea in secondary mathematics is algorithms. According to Mingus
and Grassl (1998), the “dominance of algorithmic procedures has caused many people to define knowledge in mathematics as competence in executing prescribed algorithms” (32). Usiskin (1998) breaks this down further by classifying the typical algorithms used in school mathematics into three categories. They are arithmetic algorithms, algebra and calculus algorithms, and drawing algorithms such as graphing and geometric constructions. Nevertheless, “current reform movements are de-emphasizing the importance of algorithms in favor of problem-solving approaches, the conceptualization of mathematical processes, and applications of mathematics in real-world situations” (Mingus and Grassl, 1998, 32). This is carried out primarily through algorithmic and recursive thinking. Polya’s four-step model is cited as an example of algorithmic thinking. It is summarized as understand the problem; devise a plan; implement the plan; extend the problem. Recursive thinking, on the other hand, is distinguished as iterative, self-referential, and never ending. Students need to have the chance to implement both types of thinking in creating their own rules to describe patterns and relationships that are found in problem solving (Curio and Schwartz, 1998). In section three of this paper, students will have the opportunity to practice both types of thinking as they examine the recursion found in fractals and chaos and model population growth.

A somewhat related approach to the reform movement cited above is a proposal to follow the work of fractal geometers (Simmt, 1998). “Rather than use algorithms simply to determine a result, these mathematicians use computer algorithms to generate objects, such as the Mandelbrot Set, that are of mathematical interest” (274). Simmt continues on to explain that the object then becomes the focus of investigation. By applying algorithms in this manner, students have a chance to work with fractal geometry and iteration. It also allows students to make connections between topics such as “measurement, sequences and series, geometry and limits” (277). Simmt is not alone, however, in acknowledging the continued necessity to use algorithms in traditional
Recursion is presented in the preceding paragraphs as an alternative approach to teaching algorithms. The standard notation used in the recursive process can confuse students (Hart, 1998; Franzblau and Warner, 2001). In Franzblau and Warner’s study students were given the recursive formula

\[ F_n = F_{n-1} + 3 \]  

Several misinterpretations were documented. Some students thought \( F_{n-1} \) “meant take the answer and subtract 1” rather than use the previous answer (191). These same students believed \( F_{n+1} \) meant add one to the answer. Yet, others confused \( F_{n+1} \) with its functional counterpart \( F(n+1) \). Mistakes were also attributed to sloppiness after the students were interviewed.

There are two primary schools of thought on how to deal with this issue. Hart (1998) claims that the formal subscript notation can prevent students from understanding basic recursion. He recommends replacing subscript notation with “NOW and NEXT” until “students have a solid conceptual understanding of recursion” (264,265). Nevertheless, Franzblau and Warner (2001) are proponents of using formal subscript notation from the start. Their study shows that the use of “ANSWER and PREVIOUS or NEXT and NOW may actually be a barrier” to learning standard notation (197). Furthermore, students who do not understand the notation generally ignore it until they feel comfortable with it. Franzblau and Warner have two recommendations for teachers. Model formal subscript notation from the start, but do not initially focus on it. Secondly, they found that students who could effectively describe patterns in words were most likely to use subscript notation immediately. Hence, educators should require students to write pattern descriptions. There is one similarity present in the above research. Hart (1998) and Franzblau and Warner (2001) agree that learning formal subscript notation is a long-term process regardless of the method that is chosen. Nonetheless, there is no clear answer to teaching recursion’s standard notation.
The majority of the research presented to this point leads directly to the Algebra Standard found in the *Principles and Standards for School Mathematics* (2000). It states:

Instructional programs from prekindergarten through grade 12 should enable students to-

- understand patterns, relations, and functions;
- represent and analyze mathematical situations and structures using algebraic symbols;
- use mathematical models to represent and understand quantitative relationships;
- analyze change in various contexts (32).

Technology may be able to assist even more in the implementation of the aforementioned standard. President of the NCTM, Cathy Seeley (2004), says that graphing calculators have become “essential components of an effective mathematics program” that students need to be able to use (25). She does not address the issue of programming.

Mayer, Dyck, and Vilbers (1986) do address programming in “Learning to Program and Learning to Think: What’s the Connection?”. They conducted a study in which 111 computer naïve college students took part. Neither group consisted of engineering or programming majors. Fifty-seven students were taught BASIC. After the pretest and posttest scores were analyzed, the BASIC group gained substantially more on two skills: “problem translation (as measured by word problem translation and word problem solution) and procedure comprehension (as measured by following procedures and following directions)” (607). Mayer, Dyck, and Vilbers conclude that programming “can have positive effects on thinking skills that are directly related to the language to be learned” (609).

Today, programming can be done on graphing calculators. In *Micromath*, Alan Grahm
(2003) states “the graphics calculator’s programming language is essentially BASIC” (41). He has found that “using algebra is almost unavoidable when writing even a simple program” (41). Consequently, students both learn and apply algebra in the context of programming.

One final piece of research examines the four mathematical learning styles that have been identified (Strong, Thomas, Perini, and Silver; 2004). Students who are best characterized as having the mastery style benefit most from repetitive practice and working step-by-step. Other students are categorized with an understanding style and view mathematics conceptually while looking for the reasoning behind it. Still, students with interpersonal styles thrive on application and cooperative learning. Yet, students with self-expressive styles are problem solvers who need to investigate and visualize. Therefore, teachers should attempt to accommodate the above styles by using a variety of strategies that work both to the strengths of students while forcing them to improve on their weaknesses. In part three, these styles will be reconciled by approaching fractal related activities algebraically, numerically, and graphically on both an individual and cooperative basis that involve repetitive practice, problem solving, and application.

The reviewed literature strongly supports approaching the study of mathematics from multiple perspectives. Educators must utilize algebraic, numeric, and graphic techniques in order to make the study of functions more meaningful. At the same time, algorithmic procedures must be de-emphasized while algorithmic and recursive thinking is applied to problem solving. Technology, namely graphing calculators at the high school level, may be able to help in achieving the above goals. In fact, technological proficiency is itself a standard to be reached prior to students being considered mathematically mature. Only after these steps are taken will students enter calculus with the prerequisite knowledge necessary to be successful.

In the next section, the Mandelbrot Set is a by-product of the technological advances that have taken place within the last fifty years. The computer’s ability to manipulate the large
amount of data necessary to produce the recursively-defined fractal is a recent development. The presentation of the Mandelbrot Set will remain true to the previously cited research in that it will be investigated algebraically, numerically, and graphically.
Part 2

Mandelbrot Set
Benoit Mandelbrot, the father of fractal geometry, discovered a fractal later to be known as the Mandelbrot Set (M-Set) in 1980. From all indications, the M-Set has been studied intensively ever since. One text, Peak and Frame’s *Chaos Under Control: The Art and Science of Complexity*, introduced topics related to chaos and fractals for undergraduate mathematics students and opened the door to the realm of possible applications for the M-Set in secondary mathematics.

The Mandelbrot Set is pictured below. First and foremost, it is a fractal. The property of self-similarity is exhibited on the boundary where many smaller copies of the M-Set are found, although, they are somewhat distorted. Distortion is common among fractals that

<http://fract.ygingras.net/grey/-0.65000/0.00000/0.25>
occur in nature, which lack the manufactured precision of other fractals such as Sierpinski’s Triangle or the Cantor Dust. Nevertheless, this lack of precision is far outweighed by the uniqueness of the M-Set’s Elephant and Seahorse Valleys. These are named for the shapes that are seen upon enlargement. While the physical appearance of the M-Set may be impressive, it is nothing compared to the mathematics from which it is generated.

“The Mandelbrot Set $\mu$ for the quadratic $f_c(z) = z^2 + c$ is defined as the collection of all $c \in \mathbb{C}$ (where $\mathbb{C}$ is the complex plane) for which the orbit of the point $0$ is bounded, that is,

$$\mu = \{ c \in \mathbb{C} : \{ f_c^{(n)}(0) \}_{n=0}^\infty \text{ is bounded} \}$$

(Crownover, 1995, 209).

More specifically, the distance between a point and the origin may be no greater than two in order to be included in the M-Set.

Theorem “If $|c| > 2$ and $|z| \geq |c|$, then the orbit of $z$ escapes to $\infty$.”

(210-11)

Proof:

Let $|c| = 2 + p$ where $p > 0$

$$|z_n^2| = |z_n^2 + c - c|$$

$$|z_n^2 + c - c| \leq |z_n^2 + c| + |c| \quad \text{Triangle Inequality}$$

$$|z_n^2| \leq |z_n^2 + c| + |c|$$

$$|z_n^2 + c| \geq |z_n^2| - |c|$$

$$|z_n^2 + c| \geq |z_n^2| - |z_n| \quad |z_n| \geq |c|$$

$$|z_n^2 + c| \geq |z_n| (|z_n| - 1)$$

$$|z_n^2 + c| \geq |z_n| (|c| - 1) \quad |z_n| \geq |c|$$

$$|z_n^2 + c| \geq |z_n| (2 + p - 1) \quad |c| = 2 + p$$

$$|z_n^2 + c| \geq |z_n| (p + 1)$$

Iterating $|z_{n+1}| \geq |z_n| (p + 1)$

$$|z_{n+2}| \geq |z_n| (p + 1)^2 \quad |z_{n+3}| \geq |z_n| (p + 1)^3$$

$p > 0$
\[ |z_{n+k}| \geq |z_n| (p+1)^k \]

Therefore, the orbit of \(z\) approaches infinity as \(n\) approaches infinity (Crownover, 1995).

The Mandelbrot Set may also be generated in the real number system by

\[ x_{n+1} = x_n^2 - y_n^2 + a \]
\[ y_{n+1} = 2x_n y_n + b \]

where \(x_0 = y_0 = 0\) with an orbit of 0. This is essentially the same as the complex definition.

Consider the complex \(z\)-plane. The real \(z\) are on the \(x\)-axis while the imaginary are on the \(y\)-axis.

So for any complex number

\[ z = x + yi \]
\[ z_n = x_n + y_n i \]
\[ z_{n+1} = z_n^2 + c \quad \text{where} \quad c = a + bi \]
\[ z_{n+1} = (x_n + y_n i)^2 + a + bi \]
\[ = x_n^2 + 2i x_n y_n + i^2 y_n^2 + a + bi \]
\[ = x_n^2 - y_n^2 + a + 2x_n y_n i + bi \]
\[ = x_n^2 - y_n^2 + a + (2x_n y_n + b)i \]
\[ = x_{n+1} + y_{n+1} \cdot i \]

(Frame and Peak, 1994)

Hence, the above two methods of generating the M-Set will be used interchangeably throughout the rest of this paper.

The mathematics involved in the M-Set is also readily apparent in a more concrete manner. For example, periodic cycles are found throughout the M-Set. These cycles are defined such that for some \(p > 0\), \(x_0 = x_p\) (Peitgen, Jurgens, and Saupe, 1992). In other words, if a bud is a 4-cycle (\(p = 4\)), the values will repeat after four complete iterations. “The antenna structure attached to each bud attached to the main cardioid codes the period of the attractor to which the
orbit of 0 converges for any (a,b)-value taken from the body of that bud" (Frame and Peak, 1994, 253). This is illustrated in Figure 1 below. Note how the bud labeled D has the numbers 1, 2, and 3 near its antennas. This is one way of knowing the bud is a 3-cycle. Later, the
calculations that support D will additionally show a repeating pattern of groups of three.

Another item that is evident is that the regions labeled D, E, and F appear twice. That is because the horizontal line (a-axis) that goes through the cleft of the cardioid serves as the line of symmetry. The portion of the M-Set above this line is known as the Northern Principal Sequence while the portion below it is the Southern Principal Sequence. The corresponding buds in both sequences have the same cycle.

In Appendix A, six points are used to illustrate the cyclic behavior of the M-Set. Using the definition from the real number system,

\[ x_{n+1} = x_n^2 - y_n^2 + a \quad \text{and} \]
\[ y_{n+1} = 2x_ny_n + b \]

where \( x_0 = y_0 = 0 \), Microsoft Excel was used to find twenty-five thousand iterations of each point. The final thirty iterations are included in Appendix A. The point A was found using \( a = .25 \) and \( b = 0 \). (Note: The (a,b) chosen for point A and the following Excel calculations represent a real ordered pair \((x,y)\), not the complex number \(c\)). The \( a, b, \) and distance columns converge to \(.5, 0, \) and \(.5 \) respectively. Since this occurs, it is said that \(.25,0\) with an orbit of \(0 (x_0 = y_0 = 0)\) converges to a fixed point. The intercept, \(.25,0\) was chosen because of its convenience in illustrating the proof of convergence.

\[ x_{n+1} = x_n^2 - y_n^2 + a = x_n^2 - 0^2 + .25 = x_n^2 + .25 \]
\[ y_{n+1} = 2x_ny_n + b = 2x_n(0) + 0 = 0 \]

\(.25,0\) converges to a fixed point.

\[ x_{n+1} = x_n \quad \text{and} \quad y_{n+1} = y_n \]
\[ x_{n+1} = x_n^2 + \frac{1}{4} \]
\[ x_n = x_n^2 + \frac{1}{4} \quad \text{by substitution} \]
\[ 0 = x_n^2 - x_n + \frac{1}{4} \]
\[ 0 = (x_n - \frac{1}{2}) (x_n - \frac{1}{2}) \]
\[ x_n = \frac{1}{2} \]

The solution is said to attract or converge to a fixed point if \( | \text{derivative at the fixed point} | \leq 1 \) and to repel if \( | \text{derivative at the fixed point} | > 1 \).

If \( x_{n+1} = x_n^2 + \frac{1}{4} \)

Let \( f(x) = x^2 + \frac{1}{4} \)

\[ f'(x) = 2x \]

\[ f'(\frac{1}{2}) = 2(\frac{1}{2}) = 1 \]

Therefore, \( x = \frac{1}{2} \) is a fixed point that attracts.

The next point, B, is located in the head of the M-Set. The list of iterations shows that approximately the same two values keep repeating in each column. Thus, each point in the head is a 2-cycle. A similar repeating pattern is also seen on the Excel calculation pages for points C, D, E, F, and G. (Additional convergence proofs are also included for B, C, and G.) However, C and D are 3-cycles, E and G are 4-cycles, and F is a 5-cycle. Note that the \((a,b)\) of D, E, and F were deliberately chosen by magnifying the Mandelbrot Set at [http://www.softlab.ece.ntua.gr/miscellaneous/mandel/mandel.html](http://www.softlab.ece.ntua.gr/miscellaneous/mandel/mandel.html). Although the cyclic behavior of E and F can still be found by counting the antennas in Figure 1 as previously stated, this method is severely limited by the clarity of the given diagram. Iterations, on the other hand, will always work. Furthermore, computers and graphing calculators (Appendix A) can be programmed to handle the necessary calculations.

**The Farey Sequence**

The Farey sequence is another method by which the periodic cycle of a bud may be found. The only significant constraint on this sequence is that the two starting buds must have consecutive cycles. The consecutive cycles of the Northern Principal Sequence (NPS) are labeled on the next page. Informally, the Farey sequence states the smallest periodic cycle
between two consecutive buds may be calculated by summing their cycles. More specifically, “two given buds of periods \( p \) and \( q \) at the cardioid determine the period of the largest bud in between them as \( p + q \)” (Peitgen, Jurgens, and Saupe, 1992, 439). These seemingly contradictory definitions are reconciled when one understands the cycle size and physical size of a bud are inversely related. Examination of the NPS above shows the physical size of a bud decreases as the cycle increases.

Recall that \( B \) is a 2-cycle and \( D \) is a 3-cycle. Scrutinize the region between \( B \) and \( D \) either on the top half of the M-Set or the bottom half. According to the Farey sequence, the smallest cycle between \( B \) and \( D \) is \( 2 + 3 = 5 \). This has been previously identified as letter \( F \). In a manner of speaking, the Farey sequence works similar to Pascal’s Triangle. Consider the following diagram and associated picture (for visual reference only). To find the smallest cycle of a bud between \( B \) and \( F \), add the two and the five to get seven. This pattern is also found between letters \( D \) and \( E \). Remember \( D \) is a 3-cycle and \( E \) is a 4-cycle.
Thus, the smallest cycle that can be found is $3 + 4 = 7$. The Farey sequence may also be applied to the buds on the buds.

Each bud on the Mandelbrot Set additionally contributes a unique arrangement of its own. First of all, recall that the Northern Principal Sequence (NPS) is made up of buds from left to right of consecutive cycle sizes $2$ (B), $3$ (D), $4$ (E), $5$, $6$, $7$, and so forth. The head, B, is a 2-cycle and its largest bud is found on the left side, which is a 4-cycle. Proceeding either clockwise or counterclockwise, the next smaller bud is a 6-cycle. The next smaller bud is an 8-cycle and so on. These cycles are obtained by multiplying the cycle size of the head by the consecutive cycle sizes from the cardioid (Peak and Frame, 1994).

\[
\begin{align*}
2 \cdot 2 & \quad 2 \cdot 3 & \quad 2 \cdot 4 & \quad 2 \cdot 5 & \quad \cdots \\
4 & \quad 6 & \quad 8 & \quad 10 & \quad \cdots
\end{align*}
\]

The same idea should be applied to the 3-cycle D pictured above. Its largest bud is found at the
top (NPS). This is the dividing point. It is also a 6-cycle. Proceeding either clockwise or
counterclockwise in the same fashion as with the head yields cycles of the following sizes.

\[
\begin{array}{cccccc}
3 \cdot 2 & 3 \cdot 3 & 3 \cdot 4 & 3 \cdot 5 & \cdots \\
6 & 9 & 12 & 15 & \cdots \\
\end{array}
\]

This pattern generalizes to work for all buds. The 6-cycle bud attached at the top of D has buds of sizes

\[
\begin{array}{cccccc}
6 \cdot 2 & 6 \cdot 3 & 6 \cdot 4 & 6 \cdot 5 & \cdots \\
12 & 15 & 24 & 30 & \cdots \\
\end{array}
\]

The Farey sequence may then be applied to show that the smallest possible cycle or period of the largest bud that exists between the 12 and 15-cycles above is a 27 cycle.

**Bifurcation Diagrams and Logistic Functions**

The cyclic behavior of the Mandelbrot Set is also seen in its bifurcation diagram. Bifurcation diagrams are used to give a “summary of the dynamics of the entire family” including the “transition to chaos” (Devaney, 1992, 82). In this case, the M-Set’s bifurcation diagram is actually a transformed version of the logistic function’s bifurcation map. Recall the M-Set is defined by

\[
x_{n+1} = x_n^2 - y_n^2 + a \text{ and } y_{n+1} = 2x_n y_n + b
\]

and the logistic function as

\[
x'_{n+1} = sx'_{n} (1-x'_{n}).
\]

where “ ‘ ’ ” is used because no connection has yet been established between the maps (Frame and Peak, 1994). If \( b = 0 \) and \( y_n = 0 \) then

\[
x_{n+1} = x_n^2 + a
\]

A transformation connecting \( x_n \) to \( x'_{n} \) is needed. Let
(1) \( x_n = Ax_n + B. \)

Substitute and solve for \( x_{n+1}. \)

\[
Ax_{n+1} + B = (Ax_n + B)^2 + a
\]

\[
Ax_{n+1} + B = A^2x_n^2 + 2ABx_n' + B^2 + a
\]

\[
Ax_{n+1} = A^2x_n^2 + 2ABx_n' + B^2 - B + a
\]

(2) \( x_{n+1} = Ax_{n}^2 + 2Bx_n' + (B^2 - B + a)/A \)

Simplify the logistic function.

\[
x_{n+1} = sx_n(1-x_n)
\]

\[
x_{n+1} = sx_n - sx_n^2
\]

(3) \( x_{n+1} = -sx_n^2 + sx_n' \)

Equations (2) and (3) are equivalent so

\[ A = -s \text{ and } \]

\[ 2B = s \text{ so } \]

\[ B = s/2 \text{ and } \]

\[ (B^2 - B + a)/A = 0. \]

By substitution,

\[
((s/2)^2 - s/2 + a)/(-s) = 0
\]

\[ s^2/4 - s/2 + a = 0 \]

\[ s^2 - 2s + 4a = 0 \]

\[ 4a = 2s - s^2 \]

(4) \( a = s/2 - s^2/4. \)

Substitute \( A \) and \( B \) into (1).

(5) \( x_n = -sx_n' + s/2 \) \hspace{1cm} (Frame and Peak, 1994).
Therefore, the bifurcation maps of the M-Set and logistic function are the same if equations (4) and (5) are true.

The bifurcation diagrams of the Mandelbrot Set and the logistic function are found in Figures 2 and 3. The diagrams illustrate the characteristic of period doubling. In Figure 2, several M-Set intervals that correspond to the bifurcation diagram are marked with their associated cycles. The 1-cycle is the fixed point behavior presented earlier. Starting at $c = .25$, there are cycles or periods of sizes 1, 2, 4, and a glimpse of 8 before the midget (3-cycle) on the tendril is seen. These cycles correspond to the number of branches found in the bifurcation diagram in Figure 2. This period doubling behavior is readily apparent in Figure 3 simply by counting the branches. The similarities between the M-Set and logistics function continue as convergence is examined.

![At left: Figure 2: M-Set Bifurcation](image1)

Above: Figure 3: Logistic Bifurcation from [http://en.wikipedia.org/wiki/Bifurcation_diagram](http://en.wikipedia.org/wiki/Bifurcation_diagram)
Feigenbaum Constant

The bifurcation diagrams may be used to find the Feigenbaum constant, δ. This constant is defined as the limit of “the ratios of the lengths of successive intervals between bifurcation points” where as the number of bifurcations, n, approach infinity

\[
\delta = \lim_{n \to \infty} \frac{(c_n - c_{n-1})}{(c_{n+1} - c_n)} \approx 4.669162
\]

(Crownover, 1994, 142).

Because of the difficulty in finding the bifurcation points, \(c_n\), the Feigenbaum constant is often given by

\[
\delta = \lim_{n \to \infty} \frac{(c^*_n - c^*_{n-1})}{(c^*_{n+1} - c^*_n)}
\]

(143).

The point \(c^*_n\) may be found between any pair of consecutive bifurcation points \(c_n\) and \(c_{n+1}\) “where there is a period 2\(^n\) orbit that is superattracting” (143). A point is said to be superattracting if its numerical derivative equals zero. These superattracting points may then be calculated using Newton’s Method. The resulting Feigenbaum constant allows for predictions of future bifurcations. The logistic function and M-Set do, however, converge on this constant at different points.

The place at which the Feigenbaum constant is found in Figures 2 and 3 is referred to as the Feigenbaum point. On the logistics bifurcation diagram, this is reached at approximately \(x = 3.57\). On the M-Set, the Feigenbaum point is approximately \(c = -1.401155\) (Crownover, 1994). To the right of this \(c\) value, period doubling takes place. To the left of it, chaos reigns. The 3-cycle midget is located in that chaotic region. Moreover, a similar split is present in Figure 3 around 3.57. The only difference is that period doubling is to the left of the Feigenbaum point as chaos takes over to the right of it. The bifurcation diagrams and long-term behavior reveal several common characteristics between the M-Set and logistic function.
The Cardioid

A unique feature of the Mandelbrot Set is the cardioid shape when graphed in polar form.

The following provides the derivation of the explicit formula for the cardioid’s boundary.

\[ z = z^2 + c \quad \text{c is Complex} \]

\[ z' = 2z \]

There are three possible cases.

<table>
<thead>
<tr>
<th>Case 1</th>
<th>Case 2</th>
<th>Case 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>(</td>
<td>2z</td>
<td>&gt; 1)</td>
</tr>
<tr>
<td>fixed point repels</td>
<td>fixed point attracts</td>
<td>fixed point is neutral</td>
</tr>
<tr>
<td>points are outside</td>
<td>points are inside</td>
<td>points make up boundary</td>
</tr>
<tr>
<td>of cardioid</td>
<td>of cardioid</td>
<td>of cardioid</td>
</tr>
</tbody>
</table>

since \(|2z| = 1\)

we know \(2z = e^{i\theta}\) so \(z = \frac{1}{2}e^{i\theta}\)

since \(z = z^2 + c\)

\[ z^2 - z + c = 0 \]

By substitution \((\frac{1}{2}e^{i\theta})^2 - \frac{1}{2}e^{i\theta} + c = 0\)

\[ \frac{1}{4}e^{2i\theta} - \frac{1}{2}e^{i\theta} + c = 0 \]

\[ c = \frac{1}{2}e^{i\theta} - \frac{1}{4}e^{2i\theta} \] complex equation of cardioid

By substitution \(c = \frac{1}{2}(\cos \theta + isin \theta) - \frac{1}{4}(\cos 2\theta + isin 2\theta)\)

\(c = \frac{1}{2}\cos \theta + \frac{1}{2}isin \theta - \frac{1}{4}\cos 2\theta - \frac{1}{4}isin 2\theta\)

\(c = \frac{1}{2}\cos \theta - \frac{1}{4}\cos 2\theta + (\frac{1}{2}sin \theta - \frac{1}{4}sin 2\theta)i\)

\(c = x + yi\)

so \(x = \frac{1}{2}\cos \theta - \frac{1}{4}\cos 2\theta\)

\(y = \frac{1}{2}\sin \theta - \frac{1}{4}\sin 2\theta\)

(Devaney, 1992; Peitgen, Jurgens, and Saupe, 1992)
For each $\theta$, $0 \leq \theta \leq 2\pi$, these two equations produce a point on the boundary of the cardioid.

On the whole, the cardioid’s boundary is yet another piece of the Mandelbrot puzzle. This puzzle is traditionally colored black because that is how the M-Set was first viewed. The points that escape to infinity are usually color-coded according to their rate of escape. These colorful pictures can be found in abundance on the internet. In fact, those pictures are what originally provided the motivation to choose the Mandelbrot Set as a topic of study. See Appendix A for an example.

At its most basic level, the Mandelbrot Set is a fractal formed through the process of iteration. In the next section, students will study three less complex fractals constructed in a similar manner. Students will also study Newton’s Method, an iterative procedure used to find the solution of an equation. Furthermore, chaos exhibited possibly best in the M-Set’s bifurcation diagram will be illustrated in both the Newton’s Method and population growth activities. The population growth activity will additionally be found to have its origins in the same logistic function that played such an integral role in the Mandelbrot Set discussion. Lastly, the research supported approach of studying mathematics from multiple perspectives that was used to examine the Mandelbrot Set will be implemented in the student activities.
Part 3

Student Component
The problem solving technique, involving algorithmic and recursive thinking, applied to the Mandelbrot Set is utilized in the classroom in this section. This is achieved through a series of lessons that are split into three activities. The first has the overall purpose of beginning the school year using fractals to introduce the concept of limits in calculus. The second activity should occur during the second quarter of calculus when the connection between Newton’s Method and chaos is possible. The last activity is for use in pre-calculus during the second quarter. Logistic functions will be used to model population growth with a twist that involves chaos.

The ideas behind the three activities come from the Mathematics Teacher. The limit lessons are a compilation of several articles from geometry reconstructed to lead into calculus. The remaining activities have primary sources. The second activity is based upon “Black Dot: Newton’s Method and a Simple One-Dimensional Fractal” by Tony J. Fischer. Meanwhile, the activity involving logistic functions closely resembles “Chaotic Behavior in the Classroom” by Robert C. Iovinelli. All the activities are approached graphically, numerically, and analytically. Furthermore, each also contains a technology component.

**Limits – Calculus – Appendix B**

The limits activity studies what happens to three fractals as they grow through the process of iteration. Students are to analyze patterns from the fractal tree and Cantor Set (Dust) before limit notation is formally introduced. The Sierpinski Gasket (Triangle) is also a fractal. Although, it is going to be approached from a different perspective – programming. After the SIERPINS program has been discussed, the students enter it into their graphing calculators. For many of the students, this is expected to be their first programming experience.
Immediately prior to the start of the school year, the decision to postpone the limits activity was made. The decision was based on the fact that the teacher has no previous experience with her incoming students, because she is new to the district. As a result, a one week intensive review of graphing relations was undertaken both with and without graphing calculators. Topics such as x and y intercepts, maximums, minimums, and piecewise functions were discussed. This is the assumed background with which the activity began.

The first day started with the students breaking into small groups to work on the fractal tree and Cantor Set worksheets. It quickly became apparent that the students possessed no previous knowledge of fractals, which was evident by the initial conversations that took place in their cooperative groups. It was clear the term fractal held no meaning. As a result, after the fractal tree was drawn, the class reconvened briefly as a whole to discuss fractal growth and self-similarity. The students then returned to small group work. Clarification was needed as to what constitutes iteration 0. Students also requested a hint on generalizing the information organized in the tables. Consequently, the entire class examined the relationship between the number of iterations and the number of new branches (5a of fractal tree) before returning to group work. The period ended with the biggest problem having been a typographical error in which inches had been inserted instead of units.

The second day revolved around the completion of the fractal tree and Cantor Set lessons. Student volunteers were used to explain the recursive patterns they had discovered. This educator provided additional guidance in the discussion that took place over what happens to the length of a single new branch as n approaches infinity (6 of fractal tree). The students understood the numerical and graphical aspects of the exercise immediately. Support was given in the form of the introduction to formal limit notation. These fractals appear to have provided an intuitive approach to limits. The students did not have difficulty making the transition from
the iteration-produced patterns of fractals to formal limit notation. Proof of this successful
transition was observed in the student led discussion of the Cantor Set lesson. The liveliest
exchange of the day was actually an algebraic question on whether $2 \cdot 2^n - 1 = 2^{n+1} - 1$. The day
ended with a brief introduction to programming during which the students accurately guessed the
purpose of about half the steps in the Sierpinski program. They had a particularly high degree of
success analyzing the first half of the program. The class then entered the first few lines of the
program into their calculators.

The final day of the activity was spent working on the Sierpinski Gasket. About half of
the time was used to debug programs. The students’ mistakes varied from missing decimal
points to skipping entire lines of code to typing in commands instead of using calculator defined
features. The students had never programmed before, however, one decided to program the
quadratic formula into his calculator after this experience. The students struggled with the last
two questions of the problem set. The class ended up drawing the first few iterations of the
Sierpinski Gasket by hand before everyone realized the corners will always remain. On the last
question involving the remaining area of the original triangle, the students had to be reminded
that points are zero dimensional before it could be completed.

Connections to Research

In the literature review, the importance of students having the opportunity to apply
algorithmic and recursive thinking in creating rules to describe patterns and relationships is cited.
The students were given the chance to analyze recursive patterns in the fractal tree and Cantor
Set lessons. A deliberate decision was also made to follow Hart’s lead in the handling of the
symbolic notation involved in recursion. This educator wanted students to focus on the new
notation in limits, not become overwhelmed in recursive notation. As a result, the students were
asked if their rules would take them from the first step to the second, from the second to the
third, and so forth. Meanwhile, multiple representations were used to facilitate the understanding of functions. For example, exercise five on the fractal tree lesson and problems two and three on the Cantor Set lesson were approached numerically and graphically. It was during the discussion over the Cantor Set that it became evident that not all of the students have the prerequisite algebra skills necessary to qualify as a mathematically mature calculus student. In fact, one student remains steadfast in his refusal to acknowledge the validity of the statement \( 2 \cdot 2^n - 1 = 2^{n+1} - 1 \). Finally, the use of graphing calculator technology had two purposes. Graphing calculators were used in the attempt to approach calculus from multiple perspectives and to lay the groundwork necessary to program Newton’s Method.

**Future**

No changes to this activity are planned. It introduced students to fractals, limits, and programming. Compared to previous classes, these students appear to have a better grasp on the concept of limit. Even though attitude is not a quantifiable measure, this class approached limits more willingly. The most pleasant surprise is the ease with which students approached programming. In the past, when teaching, the first experience the teacher usually presents to students involves the rectangle approximation method for calculating the area under a curve. The experience proved difficult at best. There is no clear reason why programming the Sierpinski Gasket should have been easier. Overall, this activity is considered successful.

**Newton’s Method – Calculus – Appendix C**

The students are expected to have a much broader knowledge base before starting this activity. Prior to studying Newton’s Method, the students will have spent close to a month studying derivatives, concluding with a section on linearizations. They will have additionally begun looking at uses of the first and second derivative in identifying maxima, minima, and
inflection points. However, the students will still have very limited programming skills. These are the experiences upon which Newton’s Method builds. (Please see Appendix C for an explanation of terms.)

Newton’s Method is a procedure used to approximate the zeros or solutions of an equation using a sequence of linearizations. A graphing calculator program will be used to investigate Newton’s Method in this activity. First and foremost, nevertheless, the students will work with the algorithm

\[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \]

by hand to understand how it works before writing a program for their calculators. Since students know they can get the zeros of many functions immediately by using the root or zero key on their calculators, they usually do not want to waste time working on the repeated iterations necessary for Newton’s Method. Therefore, this activity will take a slightly different approach by engaging students in an experiment involving chaos. The final product will be a number line colored according to the zeros on which the initial guesses converge.

On the first day of this activity, the students will examine how Newton’s Method works and will use it to find the roots of a few equations. Their assignment will consist of applying Newton’s Method to several problems and using derivatives to draw the complete graph of \( f(x) = x^3 - x \). The students will also be instructed to find the roots of this problem algebraically without using Newton’s Method.

On the second day, the students will draw on their experiences to write a graphing calculator program that may be used to calculate Newton’s Method. A sample program is provided in Appendix C. This day will be challenging. Even with extra copies of the Seirpinski Gasket program on hand, the students will not have had experience writing programs that allow the user to input information. Moreover, they will not know how to accommodate the number of
iterations chosen within the program. The best-case scenario has the students brainstorming all
the important features of the program with the teacher only contributing the appropriate
calculator commands. At the conclusion of class, the students will be split into groups and will
be told to investigate the included table using the new program. They will be finding the zero to
which each value on the table converges.

On day three, the groups will complete the worksheet that has a number line at the top.
The students will first use their findings about where convergence intervals begin and end to
place marks on the number line before finishing the page. A discussion of this activity will take
place at the end of the period or the start of day four.

**Connections to Research**

The students will be implementing algorithms in the traditional sense by using Newton’s
Method. Research agrees the use of algorithms in this manner is a continued necessity. The
students will also move beyond the informal notation proposed for initial use by Hart into the
formal standard notation that is used in recursive processes like the Mandelbrot Set found in the
last section. As research suggests, multiple opportunities will be provided for students to transfer
between algebraic, numeric, and graphic representations, including the use of a number line, as
they explore Newton’s Method. Lastly, the use of programming will continue in an effort to
obtain the positive side effects of problem translation and procedure comprehension documented
by Mayer, Dyck and Vilbers.

**Future**

The computer-generated pictures illustrating the basins of attraction of complex number
equations may provide an intriguing starting point for a new investigation. Newton’s Method
may be used to find the complex zeros, although, it is a little unusual at the high school level.
Nevertheless, it would make a solid connection to fractals.
Logistic Function – Pre-Calculus – Appendix D

Students are expected to have a working knowledge of various families of functions and their characteristics prior to beginning this activity. They must be familiar with the term end behavior and be experienced with a graphing calculator. Since this does not take place until chapter four, students will have spent much of the first semester reviewing how to solve and graph functions of various degrees including exponential and logarithmic functions. This is the foundation on which the last activity will be developed.

This activity, in which population growth will be modeled using a logistic function, is meant to supplement a very traditional pre-calculus textbook in which little development of the logistic function is provided prior to the involvement of exponential functions. This activity will start with a discussion of several possible models that may be used to represent population growth. After the choices are narrowed to the logistic function, the students will experiment with the model both numerically and graphically to determine whether it will work. They will then use graphing calculators to investigate the effect of different growth rates on the model, eventually resulting in chaos.

The first part of the activity, up to problem 13, will take place over the span of two days. At this point, the completion of Part 2 Cobwebbing and Part 3 Feigenbaum Bifurcation Diagram is unlikely. The cobwebbing lesson would provide a connection to the cyclic behavior of the Mandelbrot Set. In a like manner, the bifurcation diagram would be the same one used in the Mandelbrot discussion. The three lessons were designed to reinforce each other through the use of multiple representations. However, the implementation of the last two is doubtful due to this educator’s recent experience in the introduction of parametric graphing and many students’ unwillingness to work on their own.
Connections to Research

In choosing the logistic function to model population growth, the students will be forced to use algorithmic thinking. They will also have the opportunity to gain experience using formal recursive notation in the investigation of their model. Meanwhile, students will have multiple chances to switch between numerical and graphical representations in their pursuit to understand functions. Most of the suggested algebraic representation, nonetheless, will take place in the days following this activity.

Future

Unless another pleasant surprise occurs, such as the ease of programming in the first activity, the future will include the implementation of the cobwebbing and bifurcation lessons. They will more than likely need to be adjusted to accommodate the new textbook series that is being adopted. However, the mathematics will remain the same.

Conclusion

In reality, the part of this project that is the most exciting is the Mandelbrot Set. If possible, that is the lesson this author would like to implement in her classroom. The Mandelbrot Set provides much of what is missing in secondary school mathematics beginning with the fact that it is not a thousand years old. It was strangely invigorating to discover a new piece of mathematics that is younger than this educator while is capable of being taught at the high school level. For a change, mathematics is not about Newton, the Greeks, or Euclid.

The use of the Mandelbrot Set could also answer one of the most common criticisms leveled at secondary mathematics. That is the curriculum is a mile wide and an inch deep. Perhaps, the best example of this lack of depth is the study of the complex number system. Many students leave algebra two knowing they have studied imaginary numbers. Maybe, they
actual remember $i^2 = -1$. For the most part, however, students do not understand the complex number system.

This educator informally introduced a pre-calculus and a calculus class to the Mandelbrot Set. In pre-calculus, it was used to illustrate the composition of functions. In calculus, its colors were used to discuss the rate of change. Both groups were surprised to find out that imaginary numbers really are used for something. Furthermore, the Mandelbrot Set earned one of the most coveted but seldom heard comments in mathematics, “Neat!”.

The relationship between the Mandelbrot Set and secondary mathematics has been established beyond the complex number system. The M-Set is a fractal. It is connected to the logistics function, Newton’s Method, and chaos. Moreover, its study involves the application of algorithmic and recursive thinking while transferring between algebraic, numeric, and graphical representations. The Mandelbrot Set could easily become the foundation of a new secondary mathematics class offered after algebra two for students who do not want to pursue the pre-calculus/calculus track.
Appendix A

Mandelbrot Set
<table>
<thead>
<tr>
<th>( A )</th>
<th>( a )</th>
<th>( b )</th>
<th>Distance</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>0</td>
<td>0</td>
<td>( (a^2 + b^2)^{1/2} )</td>
</tr>
<tr>
<td>( x_{n+1} = x_n^2 - y_n^2 + a )</td>
<td>( y_{n+1} = 2x_ny_n + b )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( x_{24970} )</td>
<td>0.499959971</td>
<td>( y_{24970} )</td>
<td>0</td>
</tr>
<tr>
<td>( x_{24971} )</td>
<td>0.499959973</td>
<td>( y_{24971} )</td>
<td>0</td>
</tr>
<tr>
<td>( x_{24972} )</td>
<td>0.499959974</td>
<td>( y_{24972} )</td>
<td>0</td>
</tr>
<tr>
<td>( x_{24973} )</td>
<td>0.499959976</td>
<td>( y_{24973} )</td>
<td>0</td>
</tr>
<tr>
<td>( x_{24974} )</td>
<td>0.499959977</td>
<td>( y_{24974} )</td>
<td>0</td>
</tr>
<tr>
<td>( x_{24975} )</td>
<td>0.499959979</td>
<td>( y_{24975} )</td>
<td>0</td>
</tr>
<tr>
<td>( x_{24976} )</td>
<td>0.499959981</td>
<td>( y_{24976} )</td>
<td>0</td>
</tr>
<tr>
<td>( x_{24977} )</td>
<td>0.499959982</td>
<td>( y_{24977} )</td>
<td>0</td>
</tr>
<tr>
<td>( x_{24978} )</td>
<td>0.499959984</td>
<td>( y_{24978} )</td>
<td>0</td>
</tr>
<tr>
<td>( x_{24979} )</td>
<td>0.499959985</td>
<td>( y_{24979} )</td>
<td>0</td>
</tr>
<tr>
<td>( x_{24980} )</td>
<td>0.499959987</td>
<td>( y_{24980} )</td>
<td>0</td>
</tr>
<tr>
<td>( x_{24981} )</td>
<td>0.499959989</td>
<td>( y_{24981} )</td>
<td>0</td>
</tr>
<tr>
<td>( x_{24982} )</td>
<td>0.49995999</td>
<td>( y_{24982} )</td>
<td>0</td>
</tr>
<tr>
<td>( x_{24983} )</td>
<td>0.499959992</td>
<td>( y_{24983} )</td>
<td>0</td>
</tr>
<tr>
<td>( x_{24984} )</td>
<td>0.499959993</td>
<td>( y_{24984} )</td>
<td>0</td>
</tr>
<tr>
<td>( x_{24985} )</td>
<td>0.499959995</td>
<td>( y_{24985} )</td>
<td>0</td>
</tr>
<tr>
<td>( x_{24986} )</td>
<td>0.499959997</td>
<td>( y_{24986} )</td>
<td>0</td>
</tr>
<tr>
<td>( x_{24987} )</td>
<td>0.499959998</td>
<td>( y_{24987} )</td>
<td>0</td>
</tr>
<tr>
<td>( x_{24988} )</td>
<td>0.499959999</td>
<td>( y_{24988} )</td>
<td>0</td>
</tr>
<tr>
<td>( x_{24989} )</td>
<td>0.499960001</td>
<td>( y_{24989} )</td>
<td>0</td>
</tr>
<tr>
<td>( x_{24990} )</td>
<td>0.499960003</td>
<td>( y_{24990} )</td>
<td>0</td>
</tr>
<tr>
<td>( x_{24991} )</td>
<td>0.499960005</td>
<td>( y_{24991} )</td>
<td>0</td>
</tr>
<tr>
<td>( x_{24992} )</td>
<td>0.499960006</td>
<td>( y_{24992} )</td>
<td>0</td>
</tr>
<tr>
<td>( x_{24993} )</td>
<td>0.499960008</td>
<td>( y_{24993} )</td>
<td>0</td>
</tr>
<tr>
<td>( x_{24994} )</td>
<td>0.499960009</td>
<td>( y_{24994} )</td>
<td>0</td>
</tr>
<tr>
<td>( x_{24995} )</td>
<td>0.499960011</td>
<td>( y_{24995} )</td>
<td>0</td>
</tr>
<tr>
<td>( x_{24996} )</td>
<td>0.499960013</td>
<td>( y_{24996} )</td>
<td>0</td>
</tr>
<tr>
<td>( x_{24997} )</td>
<td>0.499960014</td>
<td>( y_{24997} )</td>
<td>0</td>
</tr>
<tr>
<td>( x_{24998} )</td>
<td>0.499960016</td>
<td>( y_{24998} )</td>
<td>0</td>
</tr>
<tr>
<td>( x_{24999} )</td>
<td>0.499960017</td>
<td>( y_{24999} )</td>
<td>0</td>
</tr>
<tr>
<td>( x_{25000} )</td>
<td><strong>0.499960019</strong></td>
<td>( y_{25000} )</td>
<td>0</td>
</tr>
</tbody>
</table>
The convergence of $B$ is a 2-cycle where $(a,b) = (-1,0)$.

$$x_{n+1} = x_n^2 - y_n^2 + a = x_n^2 - 0^2 - 1 = x_n^2 - 1$$

$$y_{n+1} = 2x_n y_n + b = 2x_n(0) + 0 = 0$$

$(-1,0)$ converges to a 2-cycle.

$$x_{n+2} = x_n \text{ and } y_{n+2} = y_n$$

$$x_{n+1} = x_n^2 - 1$$

$$x_{n+2} = (x_n^2 - 1)^2 - 1$$

$$x_n = (x_n^2 - 1)^2 - 1 \quad \text{by substitution}$$

$$0 = (x_n^2 - 1)^2 - x_n - 1$$

Using TI-89, $x_n = 0, -1, 1.618033989, -0.618033988$

Attracts if $| \text{ derivative at that point } | \leq 1$. Repels if $| \text{ derivative at that point } | > 1$.

$$x_{n+2} = (x_n^2 - 1)^2 - 1$$

Let $g(x) = (x^2 - 1)^2 - 1$

$$g'(x) = 2(x^2 - 1) \cdot 2x = 4x(x^2 - 1) = 4x^3 - 4x$$

$$g'(0) = 0 \quad \text{attracts}$$

$$g'(-1) = 0 \quad \text{attracts}$$

$$g'(1.61803398875) = 10.472135955 \text{ repels}$$

$$g'(-0.618033988) = 1.527864045 \quad \text{repels}$$

Since 0 and -1 attract, this means that they will continue to cycle on the next page. The two solutions that repel will continue to diverge from the fixed point.
<table>
<thead>
<tr>
<th>B</th>
<th>a</th>
<th>b</th>
<th>Distance</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>x_{n+1} = x_n^2 - y_n^2 + a</td>
<td>y_{n+1} = 2x_ny_n + b</td>
<td>(a^2+b^2)^{1/2}</td>
<td></td>
</tr>
<tr>
<td>X24970</td>
<td>0</td>
<td>y24970</td>
<td>0</td>
</tr>
<tr>
<td>X24971</td>
<td>-1</td>
<td>y24971</td>
<td>0</td>
</tr>
<tr>
<td>X24972</td>
<td>0</td>
<td>y24972</td>
<td>0</td>
</tr>
<tr>
<td>X24973</td>
<td>-1</td>
<td>y24973</td>
<td>0</td>
</tr>
<tr>
<td>X24974</td>
<td>0</td>
<td>y24974</td>
<td>0</td>
</tr>
<tr>
<td>X24975</td>
<td>-1</td>
<td>y24975</td>
<td>0</td>
</tr>
<tr>
<td>X24976</td>
<td>0</td>
<td>y24976</td>
<td>0</td>
</tr>
<tr>
<td>X24977</td>
<td>-1</td>
<td>y24977</td>
<td>0</td>
</tr>
<tr>
<td>X24978</td>
<td>0</td>
<td>y24978</td>
<td>0</td>
</tr>
<tr>
<td>X24979</td>
<td>-1</td>
<td>y24979</td>
<td>0</td>
</tr>
<tr>
<td>X24980</td>
<td>0</td>
<td>y24980</td>
<td>0</td>
</tr>
<tr>
<td>X24981</td>
<td>-1</td>
<td>y24981</td>
<td>0</td>
</tr>
<tr>
<td>X24982</td>
<td>0</td>
<td>y24982</td>
<td>0</td>
</tr>
<tr>
<td>X24983</td>
<td>-1</td>
<td>y24983</td>
<td>0</td>
</tr>
<tr>
<td>X24984</td>
<td>0</td>
<td>y24984</td>
<td>0</td>
</tr>
<tr>
<td>X24985</td>
<td>-1</td>
<td>y24985</td>
<td>0</td>
</tr>
<tr>
<td>X24986</td>
<td>0</td>
<td>y24986</td>
<td>0</td>
</tr>
<tr>
<td>X24987</td>
<td>-1</td>
<td>y24987</td>
<td>0</td>
</tr>
<tr>
<td>X24988</td>
<td>0</td>
<td>y24988</td>
<td>0</td>
</tr>
<tr>
<td>X24989</td>
<td>-1</td>
<td>y24989</td>
<td>0</td>
</tr>
<tr>
<td>X24990</td>
<td>0</td>
<td>y24990</td>
<td>0</td>
</tr>
<tr>
<td>X24991</td>
<td>-1</td>
<td>y24991</td>
<td>0</td>
</tr>
<tr>
<td>X24992</td>
<td>0</td>
<td>y24992</td>
<td>0</td>
</tr>
<tr>
<td>X24993</td>
<td>-1</td>
<td>y24993</td>
<td>0</td>
</tr>
<tr>
<td>X24994</td>
<td>0</td>
<td>y24994</td>
<td>0</td>
</tr>
<tr>
<td>X24995</td>
<td>-1</td>
<td>y24995</td>
<td>0</td>
</tr>
<tr>
<td>X24996</td>
<td>0</td>
<td>y24996</td>
<td>0</td>
</tr>
<tr>
<td>X24997</td>
<td>-1</td>
<td>y24997</td>
<td>0</td>
</tr>
<tr>
<td>X24998</td>
<td>0</td>
<td>y24998</td>
<td>0</td>
</tr>
<tr>
<td>X24999</td>
<td>-1</td>
<td>y24999</td>
<td>0</td>
</tr>
<tr>
<td>X25000</td>
<td>0</td>
<td>y25000</td>
<td>0</td>
</tr>
</tbody>
</table>
The convergence of \( C \) is a 3-cycle where \((a, b) = (-1.75, 0)\).

\[
x_{n+1} = x_n^2 - y_n^2 + a = x_n^2 - 0^2 - 1.75 = x_n^2 - 1.75
\]

\[
y_{n+1} = 2x_n y_n + b = 2x_n(0) + 0 = 0
\]

\((-1.75, 0)\) converges to a 3-cycle.

\[
x_{n+3} = x_n \text{ and } y_{n+3} = y_n
\]

\[
x_{n+1} = x_n^2 - 1.75
\]

\[
x_{n+2} = (x_n^2 - 1.75)^2 - 1.75
\]

\[
x_{n+3} = ((x_n^2 - 1.75)^2 - 1.75)^2 - 1.75
\]

\[
x_n = ((x_n^2 - 1.75)^2 - 1.75)^2 - 1.75 \quad \text{by substitution}
\]

\[
0 = ((x_n^2 - 1.75)^2 - 1.75)^2 - x_n - 1.75
\]

Using TI–89, \( x_n = 1.30193794065, -0.54958073375, -1.74731452869, 1.91421356237, 1.30193530428, -0.54958148188, -0.914213562373, -1.7462094325 \)

Attracts if \( | \text{derivative at that point} | \leq 1 \). Repels if \( | \text{derivative at that point} | > 1 \).

\[
x_{n+3} = (((x_n^2 - 1.75)^2 - 1.75)^2 - 1.75 = (((x_n^4 - 7/2x_n^2 + 49/4) - 7/4)^2 - 7/4
\]

\[
= (x_n^4 - 7/2x_n^2 + 21/16)^2 - 7/4 = x_n^8 - 7x_n^6 + (119/8)x_n^4 - (147/16)x_n^2 + 441/256
\]

Let \( h(x) = x^8 - 7x^6 + (119/8)x^4 - (147/16)x^2 + 441/256 \)

\[
h'(x) = 8x^7 - 42x^5 + (119/2)x^3 - (147/8)x
\]

\[
h'(1.30193794065) = .999990485486 \quad \text{attracts}
\]

\[
h'(-0.54958073375) = .99998952702 \quad \text{attracts}
\]

\[
h'(-1.74731452869) = .945556173139 \quad \text{attracts}
\]

\[
h'(1.91421356237) = 56.1126983705 \quad \text{repels}
\]

\[
h'(1.301935304288) = 1.00011293748 \quad \text{repels}
\]

\[
h'(-0.54958148188) = 1.0000002872 \quad \text{repels}
\]

\[
h'(-0.914213562373) = -6.1126983722 \quad \text{repels}
\]

\[
h'(-1.7462094325) = 1.12450973746 \quad \text{repels}
\]
<table>
<thead>
<tr>
<th>C</th>
<th>a</th>
<th>b</th>
<th>Distance</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>-1.75</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$x_{n+1} = x_n^2 - y_n^2 + a$</td>
<td>$y_{n+1} = 2x_ny_n + b$</td>
<td>$(a^2+b^2)^{1/2}$</td>
<td></td>
</tr>
<tr>
<td>$x_{24970}$</td>
<td>$-1.746981082$</td>
<td>$y_{24970}$</td>
<td>0</td>
</tr>
<tr>
<td>$x_{24971}$</td>
<td>$1.301942902$</td>
<td>$y_{24971}$</td>
<td>0</td>
</tr>
<tr>
<td>$x_{24972}$</td>
<td>$-0.05494468$</td>
<td>$y_{24972}$</td>
<td>0</td>
</tr>
<tr>
<td>$x_{24973}$</td>
<td>$-1.746981082$</td>
<td>$y_{24973}$</td>
<td>0</td>
</tr>
<tr>
<td>$x_{24974}$</td>
<td>$1.301942901$</td>
<td>$y_{24974}$</td>
<td>0</td>
</tr>
<tr>
<td>$x_{24975}$</td>
<td>$-0.054944682$</td>
<td>$y_{24975}$</td>
<td>0</td>
</tr>
<tr>
<td>$x_{24976}$</td>
<td>$-1.746981082$</td>
<td>$y_{24976}$</td>
<td>0</td>
</tr>
<tr>
<td>$x_{24977}$</td>
<td>$1.301942901$</td>
<td>$y_{24977}$</td>
<td>0</td>
</tr>
<tr>
<td>$x_{24978}$</td>
<td>$-0.054944684$</td>
<td>$y_{24978}$</td>
<td>0</td>
</tr>
<tr>
<td>$x_{24979}$</td>
<td>$-1.746981082$</td>
<td>$y_{24979}$</td>
<td>0</td>
</tr>
<tr>
<td>$x_{24980}$</td>
<td>$1.3019429$</td>
<td>$y_{24980}$</td>
<td>0</td>
</tr>
<tr>
<td>$x_{24981}$</td>
<td>$-0.054944685$</td>
<td>$y_{24981}$</td>
<td>0</td>
</tr>
<tr>
<td>$x_{24982}$</td>
<td>$-1.746981082$</td>
<td>$y_{24982}$</td>
<td>0</td>
</tr>
<tr>
<td>$x_{24983}$</td>
<td>$1.301942899$</td>
<td>$y_{24983}$</td>
<td>0</td>
</tr>
<tr>
<td>$x_{24984}$</td>
<td>$-0.054944687$</td>
<td>$y_{24984}$</td>
<td>0</td>
</tr>
<tr>
<td>$x_{24985}$</td>
<td>$-1.746981081$</td>
<td>$y_{24985}$</td>
<td>0</td>
</tr>
<tr>
<td>$x_{24986}$</td>
<td>$1.301942899$</td>
<td>$y_{24986}$</td>
<td>0</td>
</tr>
<tr>
<td>$x_{24987}$</td>
<td>$-0.054944688$</td>
<td>$y_{24987}$</td>
<td>0</td>
</tr>
<tr>
<td>$x_{24988}$</td>
<td>$-1.746981081$</td>
<td>$y_{24988}$</td>
<td>0</td>
</tr>
<tr>
<td>$x_{24989}$</td>
<td>$1.301942898$</td>
<td>$y_{24989}$</td>
<td>0</td>
</tr>
<tr>
<td>$x_{24990}$</td>
<td>$-0.05494469$</td>
<td>$y_{24990}$</td>
<td>0</td>
</tr>
<tr>
<td>$x_{24991}$</td>
<td>$-1.746981081$</td>
<td>$y_{24991}$</td>
<td>0</td>
</tr>
<tr>
<td>$x_{24992}$</td>
<td>$1.301942897$</td>
<td>$y_{24992}$</td>
<td>0</td>
</tr>
<tr>
<td>$x_{24993}$</td>
<td>$-0.054944692$</td>
<td>$y_{24993}$</td>
<td>0</td>
</tr>
<tr>
<td>$x_{24994}$</td>
<td>$-1.746981081$</td>
<td>$y_{24994}$</td>
<td>0</td>
</tr>
<tr>
<td>$x_{24995}$</td>
<td>$1.301942897$</td>
<td>$y_{24995}$</td>
<td>0</td>
</tr>
<tr>
<td>$x_{24996}$</td>
<td>$-0.054944693$</td>
<td>$y_{24996}$</td>
<td>0</td>
</tr>
<tr>
<td>$x_{24997}$</td>
<td>$-1.746981081$</td>
<td>$y_{24997}$</td>
<td>0</td>
</tr>
<tr>
<td>$x_{24998}$</td>
<td>$1.301942896$</td>
<td>$y_{24998}$</td>
<td>0</td>
</tr>
<tr>
<td>$x_{24999}$</td>
<td>$-0.054944695$</td>
<td>$y_{24999}$</td>
<td>0</td>
</tr>
<tr>
<td>$x_{25000}$</td>
<td>$-1.74698108$</td>
<td>$y_{25000}$</td>
<td>0</td>
</tr>
</tbody>
</table>
The convergence of $G$ is a 4-cycle where $(a, b) = (-1.3, 0)$.

\[
x_{n+1} = x_n^2 - y_n^2 + a = x_n^2 - 0^2 -1.3 = x_n^2 - 1.3
\]
\[
y_{n+1} = 2x_n y_n + b = 2x_n(0) + 0 = 0
\]

(-1.3,0) converges to a 4-cycle.

\[
x_{n+4} = x_n \text{ and } y_{n+4} = y_n
\]
\[
x_{n+1} = x_n^2 -1.3
\]
\[
x_{n+2} = (x_n^2 -1.3)^2 - 1.3
\]
\[
x_{n+3} = ((x_n^2 - 1.3)^2 -1.3)^2 - 1.3
\]
\[
x_{n+4} = (((x_n^2 - 1.3)^2 -1.3)^2 -1.3)^2 - 1.3
\]
\[
x_n = (((x_n^2 - 1.3)^2 -1.3)^2 -1.3)^2 - 1.3 \text{ by substitution}
\]
\[
0 = (((x_n^2 - 1.3)^2 -1.3)^2 -1.3)^2 - x_n - 1.3
\]
\[
x_n = .389018548257, .019430292333, -1.14866456912, -1.29962246374,
\]
\[
1.7449899598, .24161984871, -.744989959799, -1.24161984867
\]

Attracts if $| \text{derivative at that point} | \leq 1$. Repels if $| \text{derivative at that point} | > 1$.

\[
x_{n+4} = (((x_n^2 - 1.3)^2 -1.3)^2 -1.3)^2 - 1.3 = (((x_n^4 - 2.6x_n^2 + 1.69) -1.3)^2 -1.3)^2 -1.3
\]
\[
= ((x_n^4 - 2.6x_n^2 + .39)^2 -1.3)^2 -1.3 = (x_n^8 -5.2x_n^6 + 7.54x_n^4 -2.028 x_n^2 -1.1479)^2 - 1.3
\]
\[
p'(x) = 2(x^8 -5.2x^6 + 7.54x^4 -2.028 x^2 -1.1479)(8x^7 -31.2x^5 +30.16x^3 -4.056x)
\]
\[
p'(.389018548257) = .180542753191 \text{ attracts}
\]
\[
p'(.019430292333) = .180542753188 \text{ attracts}
\]
\[
p'(-1.14866456912) = .180542753361 \text{ attracts}
\]
\[
p'(-1.29962246374) = .180542753185 \text{ attracts}
\]
\[
p'(.17449899598) = 148.35142169 \text{ repels}
\]
\[
p'(.24161984871) = 1.44 \text{ repels} \quad p'(.-744989959799) = 4.92857831558 \text{ repels}
\]
\[
p'(-1.24161984867) = 1.44000000013 \text{ repels}
\]
<table>
<thead>
<tr>
<th>G</th>
<th>a</th>
<th>b</th>
<th>Distance</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>-1.3</td>
<td>0</td>
<td>0.389018548</td>
</tr>
</tbody>
</table>

\[ x_{n+1} = x_n^2 + a, \quad y_{n+1} = 2x_ny_n + b, \quad (a^2+b^2)^{1/2} \]

<table>
<thead>
<tr>
<th>x24970</th>
<th>0.389018548</th>
<th>y24970</th>
<th>0.389018548</th>
</tr>
</thead>
<tbody>
<tr>
<td>x24971</td>
<td>-1.148664569</td>
<td>y24971</td>
<td>1.148664569</td>
</tr>
<tr>
<td>x24972</td>
<td>0.019430292</td>
<td>y24972</td>
<td>0.019430292</td>
</tr>
<tr>
<td>x24973</td>
<td>-1.299622464</td>
<td>y24973</td>
<td>1.299622464</td>
</tr>
<tr>
<td>x24974</td>
<td>0.389018548</td>
<td>y24974</td>
<td>0.389018548</td>
</tr>
<tr>
<td>x24975</td>
<td>-1.148664569</td>
<td>y24975</td>
<td>1.148664569</td>
</tr>
<tr>
<td>x24976</td>
<td>0.019430292</td>
<td>y24976</td>
<td>0.019430292</td>
</tr>
<tr>
<td>x24977</td>
<td>-1.299622464</td>
<td>y24977</td>
<td>1.299622464</td>
</tr>
<tr>
<td>x24978</td>
<td>0.389018548</td>
<td>y24978</td>
<td>0.389018548</td>
</tr>
<tr>
<td>x24979</td>
<td>-1.148664569</td>
<td>y24979</td>
<td>1.148664569</td>
</tr>
<tr>
<td>x24980</td>
<td>0.019430292</td>
<td>y24980</td>
<td>0.019430292</td>
</tr>
<tr>
<td>x24981</td>
<td>-1.299622464</td>
<td>y24981</td>
<td>1.299622464</td>
</tr>
<tr>
<td>x24982</td>
<td>0.389018548</td>
<td>y24982</td>
<td>0.389018548</td>
</tr>
<tr>
<td>x24983</td>
<td>-1.148664569</td>
<td>y24983</td>
<td>1.148664569</td>
</tr>
<tr>
<td>x24984</td>
<td>0.019430292</td>
<td>y24984</td>
<td>0.019430292</td>
</tr>
<tr>
<td>x24985</td>
<td>-1.299622464</td>
<td>y24985</td>
<td>1.299622464</td>
</tr>
<tr>
<td>x24986</td>
<td>0.389018548</td>
<td>y24986</td>
<td>0.389018548</td>
</tr>
<tr>
<td>x24987</td>
<td>-1.148664569</td>
<td>y24987</td>
<td>1.148664569</td>
</tr>
<tr>
<td>x24988</td>
<td>0.019430292</td>
<td>y24988</td>
<td>0.019430292</td>
</tr>
<tr>
<td>x24989</td>
<td>-1.299622464</td>
<td>y24989</td>
<td>1.299622464</td>
</tr>
<tr>
<td>x24990</td>
<td>0.389018548</td>
<td>y24990</td>
<td>0.389018548</td>
</tr>
<tr>
<td>x24991</td>
<td>-1.148664569</td>
<td>y24991</td>
<td>1.148664569</td>
</tr>
<tr>
<td>x24992</td>
<td>0.019430292</td>
<td>y24992</td>
<td>0.019430292</td>
</tr>
<tr>
<td>x24993</td>
<td>-1.299622464</td>
<td>y24993</td>
<td>1.299622464</td>
</tr>
<tr>
<td>x24994</td>
<td>0.389018548</td>
<td>y24994</td>
<td>0.389018548</td>
</tr>
<tr>
<td>x24995</td>
<td>-1.148664569</td>
<td>y24995</td>
<td>1.148664569</td>
</tr>
<tr>
<td>x24996</td>
<td>0.019430292</td>
<td>y24996</td>
<td>0.019430292</td>
</tr>
<tr>
<td>x24997</td>
<td>-1.299622464</td>
<td>y24997</td>
<td>1.299622464</td>
</tr>
<tr>
<td>x24998</td>
<td>0.389018548</td>
<td>y24998</td>
<td>0.389018548</td>
</tr>
<tr>
<td>x24999</td>
<td>-1.148664569</td>
<td>y24999</td>
<td>1.148664569</td>
</tr>
<tr>
<td>x25000</td>
<td>0.019430292</td>
<td>y25000</td>
<td>0.019430292</td>
</tr>
<tr>
<td>( D )</td>
<td>( a )</td>
<td>( b )</td>
<td>Distance</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>-0.13</td>
<td>-0.74</td>
<td>( x_{n+1} = x_n^2 - y_n^2 + a )</td>
<td>( y_{n+1} = 2x_ny_n + b )</td>
</tr>
<tr>
<td>( x_{24970} )</td>
<td>( -0.13021853 )</td>
<td>( y_{24970} )</td>
<td>-0.7402484</td>
</tr>
<tr>
<td>( x_{24971} )</td>
<td>( -0.661010871 )</td>
<td>( y_{24971} )</td>
<td>-0.5472119</td>
</tr>
<tr>
<td>( x_{24972} )</td>
<td>( 0.007494534 )</td>
<td>( y_{24972} )</td>
<td>0.016574</td>
</tr>
<tr>
<td>( x_{24973} )</td>
<td>( -0.13021853 )</td>
<td>( y_{24973} )</td>
<td>-0.7402484</td>
</tr>
<tr>
<td>( x_{24974} )</td>
<td>( -0.661010871 )</td>
<td>( y_{24974} )</td>
<td>-0.5472119</td>
</tr>
<tr>
<td>( x_{24975} )</td>
<td>( 0.007494534 )</td>
<td>( y_{24975} )</td>
<td>0.016574</td>
</tr>
<tr>
<td>( x_{24976} )</td>
<td>( -0.13021853 )</td>
<td>( y_{24976} )</td>
<td>-0.7402484</td>
</tr>
<tr>
<td>( x_{24977} )</td>
<td>( -0.661010871 )</td>
<td>( y_{24977} )</td>
<td>-0.5472119</td>
</tr>
<tr>
<td>( x_{24978} )</td>
<td>( 0.007494534 )</td>
<td>( y_{24978} )</td>
<td>0.016574</td>
</tr>
<tr>
<td>( x_{24979} )</td>
<td>( -0.13021853 )</td>
<td>( y_{24979} )</td>
<td>-0.7402484</td>
</tr>
<tr>
<td>( x_{24980} )</td>
<td>( -0.661010871 )</td>
<td>( y_{24980} )</td>
<td>-0.5472119</td>
</tr>
<tr>
<td>( x_{24981} )</td>
<td>( 0.007494534 )</td>
<td>( y_{24981} )</td>
<td>0.016574</td>
</tr>
<tr>
<td>( x_{24982} )</td>
<td>( -0.13021853 )</td>
<td>( y_{24982} )</td>
<td>-0.7402484</td>
</tr>
<tr>
<td>( x_{24983} )</td>
<td>( -0.661010871 )</td>
<td>( y_{24983} )</td>
<td>-0.5472119</td>
</tr>
<tr>
<td>( x_{24984} )</td>
<td>( 0.007494534 )</td>
<td>( y_{24984} )</td>
<td>0.016574</td>
</tr>
<tr>
<td>( x_{24985} )</td>
<td>( -0.13021853 )</td>
<td>( y_{24985} )</td>
<td>-0.7402484</td>
</tr>
<tr>
<td>( x_{24986} )</td>
<td>( -0.661010871 )</td>
<td>( y_{24986} )</td>
<td>-0.5472119</td>
</tr>
<tr>
<td>( x_{24987} )</td>
<td>( 0.007494534 )</td>
<td>( y_{24987} )</td>
<td>0.016574</td>
</tr>
<tr>
<td>( x_{24988} )</td>
<td>( -0.13021853 )</td>
<td>( y_{24988} )</td>
<td>-0.7402484</td>
</tr>
<tr>
<td>( x_{24989} )</td>
<td>( -0.661010871 )</td>
<td>( y_{24989} )</td>
<td>-0.5472119</td>
</tr>
<tr>
<td>( x_{24990} )</td>
<td>( 0.007494534 )</td>
<td>( y_{24990} )</td>
<td>0.016574</td>
</tr>
<tr>
<td>( x_{24991} )</td>
<td>( -0.13021853 )</td>
<td>( y_{24991} )</td>
<td>-0.7402484</td>
</tr>
<tr>
<td>( x_{24992} )</td>
<td>( -0.661010871 )</td>
<td>( y_{24992} )</td>
<td>-0.5472119</td>
</tr>
<tr>
<td>( x_{24993} )</td>
<td>( 0.007494534 )</td>
<td>( y_{24993} )</td>
<td>0.016574</td>
</tr>
<tr>
<td>( x_{24994} )</td>
<td>( -0.13021853 )</td>
<td>( y_{24994} )</td>
<td>-0.7402484</td>
</tr>
<tr>
<td>( x_{24995} )</td>
<td>( -0.661010871 )</td>
<td>( y_{24995} )</td>
<td>-0.5472119</td>
</tr>
<tr>
<td>( x_{24996} )</td>
<td>( 0.007494534 )</td>
<td>( y_{24996} )</td>
<td>0.016574</td>
</tr>
<tr>
<td>( x_{24997} )</td>
<td>( -0.13021853 )</td>
<td>( y_{24997} )</td>
<td>-0.7402484</td>
</tr>
<tr>
<td>( x_{24998} )</td>
<td>( -0.661010871 )</td>
<td>( y_{24998} )</td>
<td>-0.5472119</td>
</tr>
<tr>
<td>( x_{24999} )</td>
<td>( 0.007494534 )</td>
<td>( y_{24999} )</td>
<td>0.016574</td>
</tr>
<tr>
<td>( x_{25000} )</td>
<td>( -0.13021853 )</td>
<td>( y_{25000} )</td>
<td>-0.7402484</td>
</tr>
<tr>
<td>E</td>
<td>a</td>
<td>b</td>
<td>Distance</td>
</tr>
<tr>
<td>-----</td>
<td>-------------</td>
<td>------------</td>
<td>------------------</td>
</tr>
<tr>
<td>0.27</td>
<td>-0.53</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$x_{n+1} = x_n^2 - y_n^2 + a$</td>
<td>$y_{n+1} = 2x_ny_n + b$</td>
<td>$(a^2+b^2)^{1/2}$</td>
</tr>
<tr>
<td>x24970</td>
<td>0.059459847</td>
<td>y24970</td>
<td>-0.8171861 0.819346436</td>
</tr>
<tr>
<td>x24971</td>
<td>-0.394257635</td>
<td>y24971</td>
<td>-0.6271795 0.7408058</td>
</tr>
<tr>
<td>x24972</td>
<td>0.032084933</td>
<td>y24972</td>
<td>-0.0354594 0.047820602</td>
</tr>
<tr>
<td>x24973</td>
<td>0.269772076</td>
<td>y24973</td>
<td>-0.5322754 0.596736206</td>
</tr>
<tr>
<td>x24974</td>
<td>0.059459847</td>
<td>y24974</td>
<td>-0.8171861 0.819346436</td>
</tr>
<tr>
<td>x24975</td>
<td>-0.394257635</td>
<td>y24975</td>
<td>-0.6271795 0.7408058</td>
</tr>
<tr>
<td>x24976</td>
<td>0.032084933</td>
<td>y24976</td>
<td>-0.0354594 0.047820602</td>
</tr>
<tr>
<td>x24977</td>
<td>0.269772076</td>
<td>y24977</td>
<td>-0.5322754 0.596736206</td>
</tr>
<tr>
<td>x24978</td>
<td>0.059459847</td>
<td>y24978</td>
<td>-0.8171861 0.819346436</td>
</tr>
<tr>
<td>x24979</td>
<td>-0.394257635</td>
<td>y24979</td>
<td>-0.6271795 0.7408058</td>
</tr>
<tr>
<td>x24980</td>
<td>0.032084933</td>
<td>y24980</td>
<td>-0.0354594 0.047820602</td>
</tr>
<tr>
<td>x24981</td>
<td>0.269772076</td>
<td>y24981</td>
<td>-0.5322754 0.596736206</td>
</tr>
<tr>
<td>x24982</td>
<td>0.059459847</td>
<td>y24982</td>
<td>-0.8171861 0.819346436</td>
</tr>
<tr>
<td>x24983</td>
<td>-0.394257635</td>
<td>y24983</td>
<td>-0.6271795 0.7408058</td>
</tr>
<tr>
<td>x24984</td>
<td>0.032084933</td>
<td>y24984</td>
<td>-0.0354594 0.047820602</td>
</tr>
<tr>
<td>x24985</td>
<td>0.269772076</td>
<td>y24985</td>
<td>-0.5322754 0.596736206</td>
</tr>
<tr>
<td>x24986</td>
<td>0.059459847</td>
<td>y24986</td>
<td>-0.8171861 0.819346436</td>
</tr>
<tr>
<td>x24987</td>
<td>-0.394257635</td>
<td>y24987</td>
<td>-0.6271795 0.7408058</td>
</tr>
<tr>
<td>x24988</td>
<td>0.032084933</td>
<td>y24988</td>
<td>-0.0354594 0.047820602</td>
</tr>
<tr>
<td>x24989</td>
<td>0.269772076</td>
<td>y24989</td>
<td>-0.5322754 0.596736206</td>
</tr>
<tr>
<td>x24990</td>
<td>0.059459847</td>
<td>y24990</td>
<td>-0.8171861 0.819346436</td>
</tr>
<tr>
<td>x24991</td>
<td>-0.394257635</td>
<td>y24991</td>
<td>-0.6271795 0.7408058</td>
</tr>
<tr>
<td>x24992</td>
<td>0.032084933</td>
<td>y24992</td>
<td>-0.0354594 0.047820602</td>
</tr>
<tr>
<td>x24993</td>
<td>0.269772076</td>
<td>y24993</td>
<td>-0.5322754 0.596736206</td>
</tr>
<tr>
<td>x24994</td>
<td>0.059459847</td>
<td>y24994</td>
<td>-0.8171861 0.819346436</td>
</tr>
<tr>
<td>x24995</td>
<td>-0.394257635</td>
<td>y24995</td>
<td>-0.6271795 0.7408058</td>
</tr>
<tr>
<td>x24996</td>
<td>0.032084933</td>
<td>y24996</td>
<td>-0.0354594 0.047820602</td>
</tr>
<tr>
<td>x24997</td>
<td>0.269772076</td>
<td>y24997</td>
<td>-0.5322754 0.596736206</td>
</tr>
<tr>
<td>x24998</td>
<td>0.059459847</td>
<td>y24998</td>
<td>-0.8171861 0.819346436</td>
</tr>
<tr>
<td>x24999</td>
<td>-0.394257635</td>
<td>y24999</td>
<td>-0.6271795 0.7408058</td>
</tr>
<tr>
<td>x25000</td>
<td>0.032084933</td>
<td>y25000</td>
<td>-0.0354594 0.047820602</td>
</tr>
<tr>
<td>( F )</td>
<td>( a )</td>
<td>( b )</td>
<td>Distance</td>
</tr>
<tr>
<td>-------</td>
<td>-----</td>
<td>-----</td>
<td>---------</td>
</tr>
<tr>
<td>(-0.51)</td>
<td>(0.56)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( x_{n+1} = x_n^2 + y_n^2 + a )</td>
<td>( y_{n+1} = 2x_n y_n + b )</td>
<td>((a^2+b^2)^{1/2})</td>
<td></td>
</tr>
<tr>
<td>( x_{24970} )</td>
<td>(0.017015069)</td>
<td>( y_{24970} )</td>
<td>(0.01479455)</td>
</tr>
<tr>
<td>( x_{24971} )</td>
<td>(-0.509929366)</td>
<td>( y_{24971} )</td>
<td>(0.56050346)</td>
</tr>
<tr>
<td>( x_{24972} )</td>
<td>(-0.564136171)</td>
<td>( y_{24972} )</td>
<td>(-0.0116343)</td>
</tr>
<tr>
<td>( x_{24973} )</td>
<td>(-0.191885739)</td>
<td>( y_{24973} )</td>
<td>(0.57312671)</td>
</tr>
<tr>
<td>( x_{24974} )</td>
<td>(-0.801654093)</td>
<td>( y_{24974} )</td>
<td>(0.34005031)</td>
</tr>
<tr>
<td>( x_{24975} )</td>
<td>(0.017015069)</td>
<td>( y_{24975} )</td>
<td>(0.01479455)</td>
</tr>
<tr>
<td>( x_{24976} )</td>
<td>(-0.509929366)</td>
<td>( y_{24976} )</td>
<td>(0.56050346)</td>
</tr>
<tr>
<td>( x_{24977} )</td>
<td>(-0.564136171)</td>
<td>( y_{24977} )</td>
<td>(-0.0116343)</td>
</tr>
<tr>
<td>( x_{24978} )</td>
<td>(-0.191885739)</td>
<td>( y_{24978} )</td>
<td>(0.57312671)</td>
</tr>
<tr>
<td>( x_{24979} )</td>
<td>(-0.801654093)</td>
<td>( y_{24979} )</td>
<td>(0.34005031)</td>
</tr>
<tr>
<td>( x_{24980} )</td>
<td>(0.017015069)</td>
<td>( y_{24980} )</td>
<td>(0.01479455)</td>
</tr>
<tr>
<td>( x_{24981} )</td>
<td>(-0.509929366)</td>
<td>( y_{24981} )</td>
<td>(0.56050346)</td>
</tr>
<tr>
<td>( x_{24982} )</td>
<td>(-0.564136171)</td>
<td>( y_{24982} )</td>
<td>(-0.0116343)</td>
</tr>
<tr>
<td>( x_{24983} )</td>
<td>(-0.191885739)</td>
<td>( y_{24983} )</td>
<td>(0.57312671)</td>
</tr>
<tr>
<td>( x_{24984} )</td>
<td>(-0.801654093)</td>
<td>( y_{24984} )</td>
<td>(0.34005031)</td>
</tr>
<tr>
<td>( x_{24985} )</td>
<td>(0.017015069)</td>
<td>( y_{24985} )</td>
<td>(0.01479455)</td>
</tr>
<tr>
<td>( x_{24986} )</td>
<td>(-0.509929366)</td>
<td>( y_{24986} )</td>
<td>(0.56050346)</td>
</tr>
<tr>
<td>( x_{24987} )</td>
<td>(-0.564136171)</td>
<td>( y_{24987} )</td>
<td>(-0.0116343)</td>
</tr>
<tr>
<td>( x_{24988} )</td>
<td>(-0.191885739)</td>
<td>( y_{24988} )</td>
<td>(0.57312671)</td>
</tr>
<tr>
<td>( x_{24989} )</td>
<td>(-0.801654093)</td>
<td>( y_{24989} )</td>
<td>(0.34005031)</td>
</tr>
<tr>
<td>( x_{24990} )</td>
<td>(0.017015069)</td>
<td>( y_{24990} )</td>
<td>(0.01479455)</td>
</tr>
<tr>
<td>( x_{24991} )</td>
<td>(-0.509929366)</td>
<td>( y_{24991} )</td>
<td>(0.56050346)</td>
</tr>
<tr>
<td>( x_{24992} )</td>
<td>(-0.564136171)</td>
<td>( y_{24992} )</td>
<td>(-0.0116343)</td>
</tr>
<tr>
<td>( x_{24993} )</td>
<td>(-0.191885739)</td>
<td>( y_{24993} )</td>
<td>(0.57312671)</td>
</tr>
<tr>
<td>( x_{24994} )</td>
<td>(-0.801654093)</td>
<td>( y_{24994} )</td>
<td>(0.34005031)</td>
</tr>
<tr>
<td>( x_{24995} )</td>
<td>(0.017015069)</td>
<td>( y_{24995} )</td>
<td>(0.01479455)</td>
</tr>
<tr>
<td>( x_{24996} )</td>
<td>(-0.509929366)</td>
<td>( y_{24996} )</td>
<td>(0.56050346)</td>
</tr>
<tr>
<td>( x_{24997} )</td>
<td>(-0.564136171)</td>
<td>( y_{24997} )</td>
<td>(-0.0116343)</td>
</tr>
<tr>
<td>( x_{24998} )</td>
<td>(-0.191885739)</td>
<td>( y_{24998} )</td>
<td>(0.57312671)</td>
</tr>
<tr>
<td>( x_{24999} )</td>
<td>(-0.801654093)</td>
<td>( y_{24999} )</td>
<td>(0.34005031)</td>
</tr>
<tr>
<td>( x_{25000} )</td>
<td>(0.017015069)</td>
<td>( y_{25000} )</td>
<td>(0.01479455)</td>
</tr>
</tbody>
</table>
The programming code necessary to produce the Mandelbrot Set on a TI-89 is given below. While many computer programs for the M-Set are available online, only one reference for graphing calculators was found. The article was written for a TI-83. That program expects the user to examine the M-Set one point at a time and to plot the results by hand. The TI-89 program that follows is set to increase by increments of two-hundredths and then to iterate 50 times before automatically plotting the point.

:mandelbr()

:Prgm

j,k real and imaginary parts of j+ki

:-3.2→xmin  
r- maximum j value

:3.2→xmax  
p- maximum k value

:.1→xscl  
x- x value

:-1.5→ymin  
y- y value

:1.5→ymax  
xn- new x

:.1→yscl  
ny- new y

:Delar j,r,k,p,x,nx,ic,y,ny,d,lc,xlist,ylist,slist  
ic- iteration counter

:ClrIO  
d- distance squared

:PlotsOff  
le-list counter

:FnOff  
xlist- stores M-Set x values

:2→r  
ylist- stores M-Set y values

:0→p  
slist- symmetry list for SPS

:0→lc

:Disp “mandelbrot set”

:Disp “be patient”

:For j,-2,r,.02
:For k,-1.5,p,.02
:0→x
:0→y
:0→ic

:Loop
:x^2–y^2+ j→nx
:2*x*y+k→ny
:2*nx+ny*ny→d
:nx→x
:ny→y
:ic+1→ic

:If d>4 or ic>50
:Exit
:EndLoop

:If d≤ 4 Then
:lc+1→lc
:x→xlist[lc]
:y→ylist[lc]
:-1*y→slist[lc]
:DispG
:PtOn xlist[lc], ylist[lc]
:PtOn xlist[lc], slist[lc]
:EndIf
:EndFor
:EndFor

:EndPrgm
Appendix B

Activity 1

Introduction to Limits

Calculus
Fractal Tree

1. At the bottom of this page, draw a vertical segment one unit long.
2. Draw two branches that are half the length of the last segment from the top of the segment.
3. Continue repeating step 2 from the ends of your new branches until you reach 1/16.
4. Fill out the following table.

<table>
<thead>
<tr>
<th>Iteration</th>
<th>Number of New Branches</th>
<th>Total Number of Branches</th>
<th>Length of 1 New Branch</th>
<th>Total Length of all Branches</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

5. What is the relationship between the number of iterations and the
   a. number of new branches?
   
   b. total number of branches?
   
   c. length of 1 new branch?
   
   d. total length of branches?

6. What happens to the length of a single new branch as n approaches infinity? Support your answer both numerically and graphically.

(Naylor, 1999)
1. The Cantor set is formed by taking one segment of length 1 unit and splitting it into thirds. The center third is then erased. Use this to fill in the following table.

<table>
<thead>
<tr>
<th>Iteration</th>
<th>Length of each New Segment</th>
<th>Number of New Segments</th>
<th>Total Length of New Segments</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. What happens to the number of new segments as \( n \) approaches infinity? Support your answer both numerically and graphically.

3. What happens to the total length of new segments as \( n \) approaches infinity? Support your answer both numerically and graphically.

(Naylor, 1999; Lornell and Westerberg, 1999; http://spanky.triumf.ca/www/fractal-info/cantor.htm)
Sierpinski Gasket (Triangle)
This fractal is started with an equilateral triangle. We then find the midpoint of each side before connecting the all three of them to form another triangle. This center triangle is then removed.

TI-84 Sierpinski Triangle Program
PROGRAM:SIERPINS
:FnOff
:ClrDraw
:PlotsOff
:AxesOff
:0→Xmin
:1→Xmax
:0→Ymin
:1→Ymax
:rand→X
:rand→Y
:For(K,1,3000)
:rand→N
:If N≤1/3
:Then
:.5X→X
:.5Y→Y
:End
:If 1/3<N and N≤2/3
:Then
:.5(.5+X)→X
:.5(1+Y)→Y
:End
:If 2/3<N
:Then
:.5(1+X)→X
:.5Y→Y
:End
:Pt-On(X,Y)
:End
Use the result of running the Sierpinski’s program to help answer the following questions.

1. What happens to the area of the original triangle as the number $K$ of iterations approaches 3000? Infinity? Explain.

2. How many points will never be removed?

3. How can the area approach ________________ in #1 if ________________ points remain in #2?

Fractal Tree

1. At the bottom of this page, draw a vertical segment one unit long.
2. Draw two branches that are half the length of the last segment from the top of the segment.
3. Continue repeating step 2 from the ends of your new branches until you reach 1/16"
4. Fill out the following table.

<table>
<thead>
<tr>
<th>Iteration</th>
<th>Number of New Branches</th>
<th>Total Number of Branches</th>
<th>Length of 1 New Branch</th>
<th>Total Length of all Branches</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1 unit</td>
<td>1 unit</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>1/2 unit</td>
<td>2 units</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>7</td>
<td>1/4 unit</td>
<td>3 units</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>15</td>
<td>1/8 unit</td>
<td>4 units</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td>31</td>
<td>1/16 unit</td>
<td>5 units</td>
</tr>
</tbody>
</table>

5. What is the relationship between the number of iterations and the
   a. number of new branches?
      \( 2^n \)
   
   b. total number of branches?
      \( 2^{n+1} - 1 \)
   
   c. length of 1 new branch?
      \( \left( \frac{1}{2} \right)^n \)
   
   d. total length of branches?
      \( n+1 \)

6. What happens to the length of a single new branch as \( n \) approaches infinity? Support your answer both algebraically and graphically.
   \[
   \lim_{n \to \infty} \left( \frac{1}{2} \right)^n = 0
   \]
The Cantor set is formed by systematically removing the middle third of each segment in each iteration. The process starts with a single segment and continues indefinitely. The number of new segments and their lengths decrease as the iterations progress, and the total length of the segments approaches zero.

### Table

<table>
<thead>
<tr>
<th>Iteration</th>
<th>Length of each New Segment</th>
<th>Number of New Segments</th>
<th>Total Length of New Segments</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1 unit</td>
<td>1</td>
<td>1 unit</td>
</tr>
<tr>
<td>1</td>
<td>( \frac{1}{3} ) unit</td>
<td>2</td>
<td>( \frac{1}{3} ) unit</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{1}{9} ) unit</td>
<td>4</td>
<td>( \frac{1}{9} ) unit</td>
</tr>
<tr>
<td>3</td>
<td>( \frac{1}{27} ) unit</td>
<td>8</td>
<td>( \frac{1}{27} ) unit</td>
</tr>
<tr>
<td>4</td>
<td>( \frac{1}{81} ) unit</td>
<td>16</td>
<td>( \frac{1}{81} ) unit</td>
</tr>
</tbody>
</table>

### Questions

2. What happens to the number of new segments as \( n \) approaches infinity? Support your answer both algebraically and graphically.

\[
\begin{align*}
\text{Number of Segments:} & \quad a^n \\
\text{Largest Segment:} & \quad (\frac{1}{3})^n
\end{align*}
\]

\[
\begin{align*}
\lim_{n \to \infty} a^n &= \infty \\
\lim_{n \to \infty} (\frac{1}{3})^n &= 0
\end{align*}
\]

3. What happens to the total length of new segments as \( n \) approaches infinity? Support your answer both algebraically and graphically.

\[
\begin{align*}
\text{Total Length:} & \quad \frac{1}{3^n} \\
\text{Largest Segment:} & \quad (\frac{1}{3})^n
\end{align*}
\]

\[
\begin{align*}
\lim_{n \to \infty} \frac{1}{3^n} &= 0
\end{align*}
\]

Sierpinski Gasket (Triangle)
This fractal is started with an equilateral triangle. We then find the midpoint of each side before connecting the all three of them to form another triangle. This center triangle is then removed.

TI-84 Sierpinski Triangle Program

PROGRAM:SIERPINS
:name
:FnOff - function yes
:CirDraw - clear draw
:PlotsOff
:AxesOff

\[
\begin{align*}
&0 \to X_{\text{min}} \\
&1 \to X_{\text{max}} \\
&0 \to Y_{\text{min}} \\
&1 \to Y_{\text{max}} \\
&\text{rand} \to X \\
&\text{rand} \to Y \\
&\text{For}(K,1,3000) \\
&\text{rand} \to N \\
&\text{if } N \leq 1/3 \\
&\text{Then} \\
&\text{.5} \times X \to X \\
&\text{.5} \times Y \to Y \\
&\text{End} \\
&\text{if } 1/3 < N \text{ and } N \leq 2/3 \\
&\text{Then} \\
&0.5(5+X) \to X \\
&0.5(1+Y) \to Y \\
&\text{End} \\
&\text{if } 2/3 < N \\
&\text{Then} \\
&0.5 \times (1+X) \to X \\
&.5Y \to Y \\
&\text{End} \\
&\text{Pt-On}(X,Y) \\
&\text{End}
\end{align*}
\]
Sierpinski Gasket (Triangle) continued

Use the result of running the Sierpinski's program to help answer the following questions.

1. What happens to the area of the original triangle as the number $K$ of iterations approaches 3000? Infinity? Explain. The area gets smaller. Area approaches 0.

2. How many points will never be removed?
   - infinitely many
   - corners of the triangle will never be removed

3. How can the area approach 0 in #1 if infinitely many points remain?
   - Points are 0 dimensional. They take up no area.
Appendix C

Activity 2

Newton’s Method

Calculus
Terms

1. derivative: rate of change or slope of a tangent line

2. linearization: process used to approximate a small portion of a complicated graph with a tangent line

3. maxima: all local and absolute maximums on a graph

4. minima: all local and absolute minimums on a graph

5. inflection point: the point(s) where the graph changes from concave up to concave down
   My calculus students like to think of it as the point that determines whether water will stay in the bowl or dump out of it.

6. zeros, roots, solutions: the point(s) where the graph crosses the x-axis

A - local minimum
B - inflection point
C - local maximum
D - zero, root, x-intercept, solution
E – tangent line through point C to the curve
Newton’s Method and Chaos

This lesson takes place after Newton’s Method has been introduced and students have had the opportunity to use it to locate roots. The following problems were assigned as part of the homework for today.

1. Show a complete graph and identify the inflection points, local maximum and minimum values, and the intervals on which the graph is rising, falling, concave up, and concave down. Do analytically, then confirm graphically.

\[ f(x) = x^3 - x \]

\([-5,5] \text{ by } [-5,5}\]
critical points: \((\sqrt{3}/3, -0.384900)\) 
\((-\sqrt{3}/3, 0.807550)\)
local maximum: \((-\sqrt{3}/3, 0.807550)\)
local minimum: \((\sqrt{3}/3, -0.384900)\)
inflection point: \((0,0)\)
rising: \((-\infty, -\sqrt{3}/3) \cup (\sqrt{3}/3, \infty)\)
falling: \([-\sqrt{3}/3, \sqrt{3}/3]\)
concave up: \((0, \infty)\)
concave down: \((-\infty, 0)\)

2. Find the roots of the above problem algebraically. Do not use Newton’s Method.

Answers: \(-1, 0, 1\)
Newton’s Method and Chaos (in class)

We are going to investigate \( f(x) = x^3 - x \) using Newton’s Method. However, to do it more easily, we are going to write a program for our calculator.

Students’ program for Newton’s should be similar to below.

\[ x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} \]

Program: NEWTONS

:ClrDraw
:PlotsOff
:Disp “GUESS FOR ROOT”
:Input X
:Disp “NUMBER ITERATIONS”
:Input I
:0 → C
:Lbl 1
:(X-(X^3-X)/(3X^2-1)) → X  

*******SEE NOTE BELOW*******

:Disp X
:C+1 → C
:If C < I
:Then
:Goto 1
:End

***Storing \( f(x) \) under Y1 allows you to use the following to replace above step***

\( (X-Y1/nDeriv(Y1,X,X)) → X \)

This step should error out on the critical points. It won’t if you use nDeriv.
Investigate $f(x) = x^3 - x$ using the Newton’s Method program. Write down whether each value converges to –1, 0, or 1. Then, find the exact point where each interval begins and ends to four decimal place. Note: You do not need to try every value below. The table is given to help organize your thoughts. Use the blank Extras column to record your interval investigations.

<p>| | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>-1.1</td>
<td>-0.66</td>
<td>-0.22</td>
<td>0.22</td>
<td>0.66</td>
<td>Extras</td>
</tr>
<tr>
<td>-1.09</td>
<td>-0.65</td>
<td>-0.21</td>
<td>0.23</td>
<td>0.67</td>
<td></td>
</tr>
<tr>
<td>-1.08</td>
<td>-0.64</td>
<td>-0.2</td>
<td>0.24</td>
<td>0.68</td>
<td></td>
</tr>
<tr>
<td>-1.07</td>
<td>-0.63</td>
<td>-0.19</td>
<td>0.25</td>
<td>0.69</td>
<td></td>
</tr>
<tr>
<td>-1.06</td>
<td>-0.62</td>
<td>-0.18</td>
<td>0.26</td>
<td>0.7</td>
<td></td>
</tr>
<tr>
<td>-1.05</td>
<td>-0.61</td>
<td>-0.17</td>
<td>0.27</td>
<td>0.71</td>
<td></td>
</tr>
<tr>
<td>-1.04</td>
<td>-0.6</td>
<td>-0.16</td>
<td>0.28</td>
<td>0.72</td>
<td></td>
</tr>
<tr>
<td>-1.03</td>
<td>-0.59</td>
<td>-0.15</td>
<td>0.29</td>
<td>0.73</td>
<td></td>
</tr>
<tr>
<td>-1.02</td>
<td>-0.58</td>
<td>-0.14</td>
<td>0.3</td>
<td>0.74</td>
<td></td>
</tr>
<tr>
<td>-1.01</td>
<td>-0.57</td>
<td>-0.13</td>
<td>0.31</td>
<td>0.75</td>
<td></td>
</tr>
<tr>
<td>-1</td>
<td>-0.56</td>
<td>-0.12</td>
<td>0.32</td>
<td>0.76</td>
<td></td>
</tr>
<tr>
<td>-0.99</td>
<td>-0.55</td>
<td>-0.11</td>
<td>0.33</td>
<td>0.77</td>
<td></td>
</tr>
<tr>
<td>-0.98</td>
<td>-0.54</td>
<td>-0.1</td>
<td>0.34</td>
<td>0.78</td>
<td></td>
</tr>
<tr>
<td>-0.97</td>
<td>-0.53</td>
<td>-0.09</td>
<td>0.35</td>
<td>0.79</td>
<td></td>
</tr>
<tr>
<td>-0.96</td>
<td>-0.52</td>
<td>-0.08</td>
<td>0.36</td>
<td>0.8</td>
<td></td>
</tr>
<tr>
<td>-0.95</td>
<td>-0.51</td>
<td>-0.07</td>
<td>0.37</td>
<td>0.81</td>
<td></td>
</tr>
<tr>
<td>-0.94</td>
<td>-0.5</td>
<td>-0.06</td>
<td>0.38</td>
<td>0.82</td>
<td></td>
</tr>
<tr>
<td>-0.93</td>
<td>-0.49</td>
<td>-0.05</td>
<td>0.39</td>
<td>0.83</td>
<td></td>
</tr>
<tr>
<td>-0.92</td>
<td>-0.48</td>
<td>-0.04</td>
<td>0.4</td>
<td>0.84</td>
<td></td>
</tr>
<tr>
<td>-0.91</td>
<td>-0.47</td>
<td>-0.03</td>
<td>0.41</td>
<td>0.85</td>
<td></td>
</tr>
<tr>
<td>-0.9</td>
<td>-0.46</td>
<td>-0.02</td>
<td>0.42</td>
<td>0.86</td>
<td></td>
</tr>
<tr>
<td>-0.89</td>
<td>-0.45</td>
<td>-0.01</td>
<td>0.43</td>
<td>0.87</td>
<td></td>
</tr>
<tr>
<td>-0.88</td>
<td>-0.44</td>
<td>0</td>
<td>0.44</td>
<td>0.88</td>
<td></td>
</tr>
<tr>
<td>-0.87</td>
<td>-0.43</td>
<td>0.01</td>
<td>0.45</td>
<td>0.89</td>
<td></td>
</tr>
<tr>
<td>-0.86</td>
<td>-0.42</td>
<td>0.02</td>
<td>0.46</td>
<td>0.9</td>
<td></td>
</tr>
<tr>
<td>-0.85</td>
<td>-0.41</td>
<td>0.03</td>
<td>0.47</td>
<td>0.91</td>
<td></td>
</tr>
<tr>
<td>-0.84</td>
<td>-0.4</td>
<td>0.04</td>
<td>0.48</td>
<td>0.92</td>
<td></td>
</tr>
<tr>
<td>-0.83</td>
<td>-0.39</td>
<td>0.05</td>
<td>0.49</td>
<td>0.93</td>
<td></td>
</tr>
<tr>
<td>-0.82</td>
<td>-0.38</td>
<td>0.06</td>
<td>0.5</td>
<td>0.94</td>
<td></td>
</tr>
<tr>
<td>-0.81</td>
<td>-0.37</td>
<td>0.07</td>
<td>0.51</td>
<td>0.95</td>
<td></td>
</tr>
<tr>
<td>-0.8</td>
<td>-0.36</td>
<td>0.08</td>
<td>0.52</td>
<td>0.96</td>
<td></td>
</tr>
<tr>
<td>-0.79</td>
<td>-0.35</td>
<td>0.09</td>
<td>0.53</td>
<td>0.97</td>
<td></td>
</tr>
<tr>
<td>-0.78</td>
<td>-0.34</td>
<td>0.1</td>
<td>0.54</td>
<td>0.98</td>
<td></td>
</tr>
<tr>
<td>-0.77</td>
<td>-0.33</td>
<td>0.11</td>
<td>0.55</td>
<td>0.99</td>
<td></td>
</tr>
<tr>
<td>-0.76</td>
<td>-0.32</td>
<td>0.12</td>
<td>0.56</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>-0.75</td>
<td>-0.31</td>
<td>0.13</td>
<td>0.57</td>
<td>1.01</td>
<td></td>
</tr>
<tr>
<td>-0.74</td>
<td>-0.3</td>
<td>0.14</td>
<td>0.58</td>
<td>1.02</td>
<td></td>
</tr>
<tr>
<td>-0.73</td>
<td>-0.29</td>
<td>0.15</td>
<td>0.59</td>
<td>1.03</td>
<td></td>
</tr>
<tr>
<td>-0.72</td>
<td>-0.28</td>
<td>0.16</td>
<td>0.6</td>
<td>1.04</td>
<td></td>
</tr>
<tr>
<td>-0.71</td>
<td>-0.27</td>
<td>0.17</td>
<td>0.61</td>
<td>1.05</td>
<td></td>
</tr>
<tr>
<td>-0.7</td>
<td>-0.26</td>
<td>0.18</td>
<td>0.62</td>
<td>1.06</td>
<td></td>
</tr>
<tr>
<td>-0.69</td>
<td>-0.25</td>
<td>0.19</td>
<td>0.63</td>
<td>1.07</td>
<td></td>
</tr>
<tr>
<td>-0.68</td>
<td>-0.24</td>
<td>0.2</td>
<td>0.64</td>
<td>1.08</td>
<td></td>
</tr>
<tr>
<td>-0.67</td>
<td>-0.23</td>
<td>0.21</td>
<td>0.65</td>
<td>1.09</td>
<td></td>
</tr>
</tbody>
</table>
1. Label the local minimum, maximum, inflection point, and roots on the number line.

2. Color the intervals that converge to
   a) –1 ___________________ (Choose and color the number line.)
   b) 0 ___________________
   c) 1 ___________________

3. What is one characteristic of fractals that is present?

4. What does chaos mean to you? What do you think chaos means in math? Put rectangles around the regions on the number line that represent chaos.

(Fischer, 2001; Vanden Bosch, 2000; Bedford, 1998)
Appendix D

Activity 3

Population Growth Modeled with Logistic Functions

Pre-Calculus
Population and Chaos

1. Are the following models realistic interpretations of population growth? Explain. Let $a =$ rate of growth.

   a) Linear Growth Model $y = ax$ Let $a = 2.5$

   \[-10,10\] by \([-10,10]\]

   b) Exponential Growth Model $y = a^x$ Let $a = 2.5$

   \([-10,10]\) by \([-5,15]\]

2. What factors should you consider when trying to predict population growth?

3. Would the following model be a better representation of population growth? Explain.

   Logistic Growth Model

   $P_{n+1} = aP_n(1-P_n)$

   $P_{n+1} =$

   $a =$

   $P_n =$

   $(1-P_n) =$
5. Use \( P_{n+1} = aP_n(1-P_n) \) to find the population for the first 15 generations. Let \( a = 2.5 \) and \( P_n = .02 \)

\[
\begin{align*}
P_{0+1} &= P_1 = \\
P_{1+1} &= P_2 = \\
P_{2+1} &= P_3 = \\
P_{3+1} &= P_4 = \\
P_{4+1} &= P_5 = \\
P_{5+1} &= P_6 = \\
P_{6+1} &= P_7 = \\
P_{7+1} &= P_8 = \\
P_{8+1} &= P_9 = \\
P_{9+1} &= P_{10} = \\
P_{10+1} &= P_{11} = \\
P_{11+1} &= P_{12} = \\
P_{12+1} &= P_{13} = \\
P_{13+1} &= P_{14} = \\
P_{14+1} &= P_{15} = 
\end{align*}
\]

6. Plot the 15 ordered pairs you just found \((n+1, P_{n+1})\) 
   \(n+1 = \) generation #  
   Example:  \(0+1 = 1^{st} \) generation

\[
\begin{array}{c|c}
 n+1 & P_{n+1} \\
 \hline 
 0+1 & P_1 \\
 1+1 & P_2 \\
 2+1 & P_3 \\
 3+1 & P_4 \\
 4+1 & P_5 \\
 5+1 & P_6 \\
 6+1 & P_7 \\
 7+1 & P_8 \\
 8+1 & P_9 \\
 9+1 & P_{10} \\
 10+1 & P_{11} \\
 11+1 & P_{12} \\
 12+1 & P_{13} \\
 13+1 & P_{14} \\
 14+1 & P_{15} \\
\end{array}
\]

7. What does the answer to #5 and the end behavior asymptote on #6 seem to conclude?
Part 1
Next, we are going use your graphing calculator to investigate what happens if we change the
growth rate. We will use the same logistic function above in the sequence mode of your
calculator.
\[ P_{n+1} = aP_n(1-P_n) \] which is equivalent to \( y = ax(1-x) \)

Calculator notation \( u(n) = Au(n-1)(1-u(n-1)) \)

**TI-83 and 84**  
*Note: \( u \) is obtained by pressing 2\(^{nd}\) and 7*

Mode  
<table>
<thead>
<tr>
<th>Normal</th>
<th>Sci Eng</th>
<th>Y =</th>
</tr>
</thead>
<tbody>
<tr>
<td>Float</td>
<td>0123456789</td>
<td>( u(n) = \text{expr} )</td>
</tr>
<tr>
<td>DegRad</td>
<td>Func Par Pol Seq</td>
<td>( u(n)=8.01 )</td>
</tr>
<tr>
<td>Connected Dot</td>
<td>Sequential Simul</td>
<td>( u(n)=0 )</td>
</tr>
<tr>
<td>Real a(+)b re(^{\circ})t</td>
<td>Full Horiz G-T</td>
<td>( u(n)=-)</td>
</tr>
</tbody>
</table>

**WINDOW**  
- \( xMin=1 \)
- \( xMax=50 \)
- \( PlotStart=1 \)
- \( PlotStep=1 \)
- \( xMin=0 \)
- \( xMax=50 \)
- \( YMin=0 \)
- \( YMax=1 \)
- \( Xscl=1 \)
- \( Yscl=1 \)

nMin and nMax tells it to evaluate 1 through 50  
Plot Start = first number to be plotted  
Plot Step = tells increment by which Plot Start increases

8. Return to home screen and store 2.9 as A then graph.  
Sketch of graph  
Table  
<table>
<thead>
<tr>
<th>( n )</th>
<th>( u(n) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>45</td>
<td>0.65418</td>
</tr>
<tr>
<td>46</td>
<td>0.65600</td>
</tr>
<tr>
<td>47</td>
<td>0.65497</td>
</tr>
<tr>
<td>48</td>
<td>0.65589</td>
</tr>
<tr>
<td>49</td>
<td>0.65582</td>
</tr>
<tr>
<td>50</td>
<td>0.65665</td>
</tr>
<tr>
<td>51</td>
<td>0.65645</td>
</tr>
</tbody>
</table>

How does this compare to your answer in number 7? Explain.

Store the following values for A, sketch a graph and list the table’s values \( 45\leq n\leq 51 \).
9. A = 3.3  
Sketch of graph  
Table  
Describe the end behavior of this sequence.
10. $A = 3.54$
Sketch of graph Table Describe the end behavior of this sequence.

Another sketch in connected mode

11. $A = 3.9$
Sketch of graph Table Describe the end behavior of this sequence.

12. What is chaos?

13. What should this mean to everyone who uses mathematical modeling to study real world issues?
ANSWER KEY – Part 1

Next, we are going use your graphing calculator to investigate what happens if we change the growth rate. We will use the same logistic function above in the sequence mode of your calculator.

\[ P_{n+1} = aP_n(1-P_n) \] which is equivalent to \[ y = ax(1-x) \]

Calculator notation \( u(n) = Au(n-1)(1-u(n-1)) \)

TI-83 and 84  Note:  \( u \) is obtained by pressing 2nd and 7

Mode

<table>
<thead>
<tr>
<th>Normal</th>
<th>Sci Eng</th>
<th>Plot 1</th>
<th>Plot 2</th>
<th>Plot 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Float</td>
<td>0123456789</td>
<td>( u(n) = 1 )</td>
<td>( u(n-1) = 0.01 )</td>
<td>( u(n-1) = 0 )</td>
</tr>
<tr>
<td>Radian</td>
<td>Degree</td>
<td>Connected Dot</td>
<td>Sequential</td>
<td>Sinul</td>
</tr>
</tbody>
</table>
| Func Par Pol Sex | | | Real a+bxt+cx^2 | Full Horiz | T-

\[ nMin = 1 \] \( nMax = 50 \)
\( PlotStart = 1 \) \( PlotStep = 1 \)
\( Xmin = 0 \) \( Xmax = 50 \)
\( Ymin = 0 \) \( Ymax = 50 \)
\( Xscl = 1 \) \( Yscl = 1 \)

8. Return to home screen and store 2.9 as A then graph.
Sketch of graph  Table

<table>
<thead>
<tr>
<th>( n )</th>
<th>( u(n) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>45</td>
<td>0.65418</td>
</tr>
<tr>
<td>46</td>
<td>0.65406</td>
</tr>
<tr>
<td>47</td>
<td>0.65375</td>
</tr>
<tr>
<td>48</td>
<td>0.65589</td>
</tr>
<tr>
<td>49</td>
<td>0.65576</td>
</tr>
<tr>
<td>50</td>
<td>0.65465</td>
</tr>
<tr>
<td>51</td>
<td>0.65465</td>
</tr>
</tbody>
</table>

How does this compare to your answer in number 7? Explain.

Store the following values for A, sketch a graph and list the table’s values 45 ≤ \( n \) ≤ 51.
9. \( A = 3.3 \)
Sketch of graph  Table

<table>
<thead>
<tr>
<th>( n )</th>
<th>( u(n) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>45</td>
<td>0.8236</td>
</tr>
<tr>
<td>46</td>
<td>0.7943</td>
</tr>
<tr>
<td>47</td>
<td>0.8236</td>
</tr>
<tr>
<td>48</td>
<td>0.7943</td>
</tr>
<tr>
<td>49</td>
<td>0.8236</td>
</tr>
<tr>
<td>50</td>
<td>0.7943</td>
</tr>
<tr>
<td>51</td>
<td>0.8236</td>
</tr>
</tbody>
</table>

Describe the end behavior of this sequence.

End behavior is two valued.

"Bifurcation"
ANSWER KEY CONTINUED – Part 1

10. \( A = 3.54 \)
    Sketch of graph
    Table
    Describe the end behavior of this sequence.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( u(n) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>45</td>
<td>.82195</td>
</tr>
<tr>
<td>46</td>
<td>.51806</td>
</tr>
<tr>
<td>47</td>
<td>.88288</td>
</tr>
<tr>
<td>48</td>
<td>.36343</td>
</tr>
<tr>
<td>49</td>
<td>.81897</td>
</tr>
<tr>
<td>50</td>
<td>.52483</td>
</tr>
<tr>
<td>51</td>
<td>.88282</td>
</tr>
</tbody>
</table>

\( n=45 \)

End behavior is four valued.

Have students change to connected mode to see oscillation.

11. \( A = 3.9 \)
    Sketch of graph
    Table
    Describe the end behavior of this sequence.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( u(n) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>45</td>
<td>.3922</td>
</tr>
<tr>
<td>46</td>
<td>.29368</td>
</tr>
<tr>
<td>47</td>
<td>.25486</td>
</tr>
<tr>
<td>48</td>
<td>.74081</td>
</tr>
<tr>
<td>49</td>
<td>.74083</td>
</tr>
<tr>
<td>50</td>
<td>.73352</td>
</tr>
<tr>
<td>51</td>
<td>.76339</td>
</tr>
</tbody>
</table>

\( n=45 \)

Fluctuates randomly
Chaos

12. What is chaos?

“property of a mathematical system where a small difference in initial conditions could result in radically different predictions”

13. What should this mean to everyone who uses mathematical modeling to study real world issues?

(Bedford, 1998; Iovinelli, 2000; Iovinelli, 1997)
Part 2

Cobwebbing

TI-83 and 84  Note: u is obtained by pressing 2^{nd} and 7

14. Store \( A = 2.9 \) Graph Trace

What happens?

You will see

\( y = ax(1-x) \)
\( y = x \)

Every two presses of right arrow is 1 more step of iteration.

Compare this to #8 in Part 1.

15. Store \( A = 3.3 \) Graph, Trace, and Sketch

What happens?

Compare this to #9 in Part 1.
16. Store A = 3.54

<table>
<thead>
<tr>
<th>Store</th>
<th>Graph, Trace, and Sketch</th>
<th>What happens?</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.9+A</td>
<td>3.3</td>
<td></td>
</tr>
<tr>
<td>3.3+A</td>
<td>3.3</td>
<td></td>
</tr>
<tr>
<td>3.54+A</td>
<td>3.54</td>
<td></td>
</tr>
</tbody>
</table>

Compare this to #10 in Part 1.

17. Store A = 3.9

<table>
<thead>
<tr>
<th>Store</th>
<th>Graph, Trace, and Sketch</th>
<th>What happens?</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.3+A</td>
<td>3.3</td>
<td></td>
</tr>
<tr>
<td>3.54+A</td>
<td>3.54</td>
<td></td>
</tr>
<tr>
<td>3.9+A</td>
<td>3.9</td>
<td></td>
</tr>
</tbody>
</table>

Compare this to #11 in Part 1.

18. What is the relationship between the activity in Part 1 and this activity in Part 2?
Part 2 – MY ANSWER KEY
Cobwebbing

TI-83 and 84  Note:  u is obtained by pressing 2nd and 7

14. Store A = 2.9 Graph Trace

\[
\begin{align*}
2.9 + A & 2.9 \\
\end{align*}
\]

You will see
\[
y = ax(1-x) \\
y = x
\]

What happens?

Compare this to #8 in Part 1.

Every two presses of right arrow is 1 more step of iteration.

converges to 1 value

15. Store A = 3.3 Graph, Trace, and Sketch

\[
\begin{align*}
2.9 + A & 2.9 \\
3.3 + A & 3.3 \\
\end{align*}
\]

What happens?

Compare this to #9 in Part 1.

Two valued

Rotates Through Two Corners

16. Two valued

Rotates Through Two Corners
Store A = 3.54 Graph, Trace, and Sketch What happens?

Compare this to #10 in Part 1.

"Four valued" - rotates through corners

17. Store A = 3.9 Graph, Trace, and Sketch What happens?

Compare this to #11 in Part 1.

chaos - cobwebs everywhere

18. What is the relationship between the activity in Part 1 and this activity in Part 2?
Part 3
Feigenbaum Bifurcation Diagram

Use Catalog (2nd 0) to find FnOff and other keys. Note: Window features are found under Vars, 1 Window

To run the program, 2nd Quit, PRGM, EXEC, Enter. You should obtain a diagram that looks like the following.

20a. Where is there only one branch in the bifurcation diagram? Compare this to 8 and 14.

20b. When does A converge to a single point?

21a. Where are there two branches in the bifurcation diagram? Compare this to 9 and 15.
21b. When is $A$ two-valued?

22a. Where are there four branches in the bifurcation diagram? Compare this to 10 and 16.

22b. When is $A$ four-valued?

23a. Where does chaos reign in the bifurcation diagram? Compare this to 11 and 17.

23b. Which $A$ values produce chaos?

(Hanna, 2001)
Part 3 – My Answer Key

Feigenbaum Bifurcation Diagram

Use Catalog ($2^{nd}$ 0) to find FnOff and other keys. Note: Window features are found under Vars, 1 Window

To run the program, $2^{nd}$ Quit, PRGM, EXEC, Enter. You should obtain a diagram that looks like the following.

20a. Where is there only one branch in the bifurcation diagram? Compare this to 8 and 14.

20b. When does $A$ converge to a single point?

$1 < A < 3$

21a. Where are there two branches in the bifurcation diagram? Compare this to 9 and 15.
21b. When is $A$ two-valued?

$$3 < A < 3.4$$

22a. Where are there four branches in the bifurcation diagram? Compare this to 10 and 16.

22b. When is $A$ four-valued?

$$3.4 < A < 3.57$$

23a. Where does chaos reign in the bifurcation diagram? Compare this to 11 and 17.

23b. Which $A$ values produce chaos?

$$A \geq 3.57$$

*** Show region of stability in midst of chaos – 3-cycle. (Hanna, 2001)
Part 4

Bibliography


