Existence of strong solution for a class of nonlinear parabolic systems

by

Kunlun Liu

A Creative Component submitted to my graduate committee
in partial fulfillment of the requirements for the degree of

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Major: Applied Mathematics

Program of Study Committee:
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DEDICATION

I would like to dedicate this dissertation to my wife Jie without her support I would not have been able to complete this work.
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CHAPTER 1. INTRODUCTION

We study the Cauchy-Dirichlet problem of strong solution for a class of nonlinear parabolic partial difference equations

\[ \frac{\partial u}{\partial t} + f(u, x, Du, D^2U) = f_0 \]  \hspace{1cm} (1.1)

\[ u(0, x) = g(x), \quad u(t; x) = h(t; x) \text{ when } x \in \partial U. \]  \hspace{1cm} (1.2)

in the domain \([0, T] \times U\), where the nonlinear function \(f\) is a polynomial of linear operators and can be represented by

\[ f(u, x, Du, D^2U) = Lu - L_1u. \]  \hspace{1cm} (1.3)

Here, \(Lu\) is a linear part of \(f\) obeying

\[ Lu := -D_i(a^{ij}(x)D_ju) + b(x)u, \]  \hspace{1cm} (1.4)

and \(L_1u\) is the high order part of \(f\) given by

\[ L_1u = \sum_{i=2}^{M} \sum_{n=1}^{N_i} \sum_{j=1}^{i} F_{i,j,n}(u). \]  \hspace{1cm} (1.5)

We assume that the Dirichlet problem of the system (A.1)-(A.2) hold following conditions:

\textbf{(A1)}, \(U\) is a bounded connected open set in \(\mathbb{R}^N\) satisfying \(C^{m+2s+2,1}\)-regularity property, and there exists a \(m + 2s + 2\)-smooth one to one transform of \(U\) onto disk:\((0,1)\), where \(\infty > m > N/2, \infty > s > 0,\) and \(N\) is a positive integer.

\textbf{(A2)}, \(a^{ij}(x) \in C^{m+2s+1,1}(U)\) and \(b(x)\) are non-negative. \(L\) is self-adjoint, bounded and strong elliptic. i.e. there exist positive number \(\gamma_1, \gamma_2,\) and a non-negative number \(\gamma_3\) such that \(a^{ij} = a^{ji},\)

\[ |a^{ij}(x)\xi_i \xi_j| \geq \gamma_1 |\xi|^2, \quad \forall x \in U, \xi \in \mathbb{R}^N, \]  \hspace{1cm} (1.6)

\[ \sum |a^{ij}(x)|^2 \leq \gamma_2, \quad \forall x \in U, \]  \hspace{1cm} (1.7)
and
\[ \infty > b(x) \geq \gamma_3 \geq 0, \quad \forall x \in U. \] (1.8)

Moreover, \( g \in H^{m+2s+1}(\mathcal{U}) \), \( h \in \mathbb{H}^{1}(0,T;H^{m+2s+1}(\partial\mathcal{U})) \cap \mathbb{H}^{0}(0,T;H^{m+2s+3}(\partial\mathcal{U})) \), and \( f_0 \in H^{m+2s}(\mathcal{U}) \) is a force independent of \( u \).

(A3), each \( F_{i,j,n} \) is a linear operator from \( \mathbb{H}^{j}(0,T;H^{a+p_{i,j,n}}(U)) \) into \( \mathbb{H}^{j}(0,T;H^{a}(U)) \) so that
\[ \|F_{i,j,n}(u)\|_{\mathbb{H}^{j}(0,T;H^{a}(U))} \leq a_{i,j,n} \|u\|_{\mathbb{H}^{j}(0,T;H^{a+p_{i,j,n}}(U))} \] (1.9)
where \( m + 2s + 1 \geq a \geq 0 \), \( s \geq j \geq 0 \), \( 1 \geq p_{i,j,n} > -\infty \), and the space \( \mathbb{H}^{j}(0,T;H^{a}(U)) \) is defined by definition 11.

In summary, the equation is of the form:
\[ \frac{\partial u}{\partial t} - D_i(a^{ij}(x)D_ju) + b(x)u - \sum_{i=2}^{M} \sum_{n=1}^{N_i} \prod_{i=1}^{M} F_{i,j,n}(u) = f_0 \] (1.10)

\( f \) is uniformly elliptic, but may not be convex or proper because of the condition (A2) and (A3). When the condition (A1) and (A2) are held, the eigenvalues of operator \( Lu \) are positive and the eigenfunctions \( \varphi_{\lambda_i} \in H^1_0(U) \cap H^{m+2s+2}(U) \) span a Hilbert space \( H(U) \) (theorem 8.37, Gilbarg and Trudinger (2001)). Moreover, \( F_{i,j,n} \) can be convolution, scaler multiplication, differential or integral operator etc. Nonlinear parabolic equation (A.1) is a general representation of KDV equation and Fokker-Plank equation etc. Here, the \( i \) is the order of the nonlinear operator \( \prod_{j=1}^{i} F_{i,j,n} \) and the \( N_i \) is the maximum number of the nonlinear operator \( \prod_{j=1}^{i} F_{i,j,n} \) with the order \( i \).

**Example 1** An typical example satisfying the above requirements is the Burger’s equation, which is given by
\[ \frac{\partial u_s}{\partial t} + \sum_{i} \frac{\partial u_s u_j}{\partial x_j} = \mu \Delta u_s \] (1.11)
where \( \mu \) is a constant. The Burger’s equation is a particular case of the Navier-Stokes equations. The linear part of (1.11) is a heat equation. The nonlinear part of (1.11) is \( \sum_{j} \frac{\partial u_s u_j}{\partial x_j} \), which is
\[ u_s \sum_{i} \frac{\partial u_j}{\partial x_j} + \sum_{i} u_j \frac{\partial u_s}{\partial x_j} \] (1.12)
Note that $u_s$ and $\frac{\partial u_s}{\partial x_i}$ can be represent by $F_{i,j,n}(u_s)$ where $F_{i,j,n}$ is a linear operator satisfying (A3). Without loss of generality, we discuss the two-dimensional Burger’s equation. Thus, (1.12) is

$$
u_s \frac{\partial u_1}{\partial x_1} + u_s \frac{\partial u_2}{\partial x_2} + u_1 \frac{\partial u_s}{\partial x_1} + u_2 \frac{\partial u_s}{\partial x_2}$$

(1.13)

where $s = 1, 2$. By using the representation of $F_{i,j,n}$, (1.13) can be reduced to

$$-\sum_{i=2}^{2} \sum_{n=1}^{4} \sum_{j=1}^{2} F_{i,j,n}(u_s)$$

where

$$F_{2,1,1}(u_s) = -u_s, \quad F_{2,2,1}(u_s) = \frac{\partial u_1}{\partial x_1},$$

$$F_{2,1,2}(u_s) = -u_s, \quad F_{2,2,2}(u_s) = \frac{\partial u_2}{\partial x_2},$$

$$F_{2,1,3}(u_s) = -u_1, \quad F_{2,2,3}(u_s) = \frac{\partial u_s}{\partial x_1},$$

$$F_{2,1,4}(u_s) = -u_2, \quad F_{2,2,4}(u_s) = \frac{\partial u_s}{\partial x_2}.$$
approach is to prove the existence of classical solutions of the Dirichlet problem in a smooth bounded domain \( U \in \mathbb{R}^N \) directly using the continuity method (Evans (1982), Krylov (1983), and Caffarelli and Huang (2003)). For this, one needs to prove a priori estimates for solutions in the space \( C^{2,\alpha}(\overline{U}) \) for some \( 0 < \alpha < 1 \). The second approach called viscosity solution method is to prove the existence of some sort of generalized solutions and then to achieve their uniqueness and regularity (Evans (1978), Crandall and Lions (1983), and Crandall et al. (1992)). A major breakthrough in the theory of viscosity solutions was made by Jensen Jensen (1988), who proved a comparison principle which turned out the uniqueness of viscosity solutions of the Dirichlet problem for

\[
f(D^2u, x) = 0
\]

at least for \( f \) independent of \( x \), but the existence of solutions of the Dirichlet problem is not proved. Indeed, our knowledge about the existence and high order regularity of fully nonlinear PDEs remains elusive, particularly for the nonlinear PDEs which are not convex or proper. A typical case is the Navier-Stokes equation. Although the existence of the weak solution (Leray (1934), Caffarelli et al. (1982)) and mild solution Kato (1984) of Navier-Stokes equation are known, the global existence of the strong solution still remains open.

The objective of this study is to prove the Dirichlet problem of the strong global solution in the space \([0, T] \times U\) for a class of nonlinear parabolic systems (A.1)-(A.2) under the condition (A1), (A2), and (A3), where the system may not be convex.

As a prelude to major theorems we show now some fundamental definition.

For a fully nonlinear partial differential equation of the form \( F(x, u, Du, D^2u) = 0 \), \( F \) is called proper if

\[
F(x, r, p, X) \leq F(x, s, p, X) \quad \text{whenever } r \leq s.
\]

\( F \) is called degenerate elliptic if

\[
F(x, r, p, X) \leq F(x, r, p, Y) \quad \text{whenever } Y \leq X.
\]

We say \( F \) is uniformly elliptic if there are constants \( \lambda, \nu > 0 \) for which

\[
\lambda \text{trace}(P) \leq F(x; r; p; X - P) - F(x; r; p; X) \leq \nu \text{trace}(P)
\]
where $\lambda$ and $\nu$ are elliptic constant.
CHAPTER 2. NOTATION AND THE MAIN IDEAS

2.1 Definition

**Definition 1** Denote $W^{m,p}(U)$ by a vector space consists of all functions $u$ such that

$$
\|u\|_{W^{m,p}(U)} := \sum_{|\alpha| \leq m} \|D^\alpha u\|_{L^p(U)} < \infty
$$

where $U \subset \mathbb{R}^N$ and $D^\alpha u$ is the weak derivative of $u$.

**Definition 2** Denote $C^{m,\lambda}(U)$ by a vector space consists of all functions $u$ such that

$$
\|u\|_{C^{m,\lambda}(U)} := \sup_{x,y \in U, x \neq y} \frac{|D^m u(x) - D^m u(y)|}{(x - y)^\lambda} < \infty
$$

where $U \subset \mathbb{R}^N$ and $D^m u$ is the strong derivative of $u$.

**Definition 3** Denote $C^j_B(U)$ by a vector space consists of all functions $u$ having bounded, continuous derivatives up to order $j$ on $\Omega$ normed by

$$
\|u\|_{C^j_B(U)} := \max_{0 \leq j \leq m} \sup_{x \in U} |D^\alpha u(x)| < \infty
$$

where $U \subset \mathbb{R}^N$ and $D^\alpha u$ is the strong derivative of $u$.

**Definition 4** Denote $\mathcal{S}^s(0,T;H^m(U))$ by a vector space consists of all functions $u$ such that

$$
\|u\|_{\mathcal{S}^s(0,T;H^m(U))} := \sum_{0 < j < s} \max_{0 \leq t \leq T} \left\| \frac{\partial^j u}{\partial t^j} \right\|_{H^m(U)} < \infty
$$

where $U \subset \mathbb{R}^N$.

**Definition 5** Denote $\mathcal{K}^s_q(0,T;U)$ by a vector space consists of all functions $u$,

$$
u \in \bigcap_{j=0}^s \mathcal{S}^{s-j}(0,T;H^{m+j}(U))$$ (2.1)
where $U \subset \mathbb{R}^N$ and $q \geq 2$. $\|u\|_{\mathcal{D}^s(0,T; \mathcal{H}^m(U))}$ is defined by

$$\|u\|_{\mathcal{D}^s(0,T; U)} = \sum_{0 < j < s} \|u\|_{\mathcal{D}^{s-j}(0,T; \mathcal{H}^{m+j}(U))}$$

**Definition 6** Let the constant $\varepsilon^*$ be arbitrary in the domain $[\frac{1}{4}, \frac{1}{2}]$, $\theta = \frac{4C_M}{\varepsilon^*}$,

$$C_{\gamma} = 2N^{m+2(s+1)} \max(1, \frac{\sqrt{T}}{\sqrt{\gamma_1}}, \min(T, \frac{\sqrt{T}}{\sqrt{\gamma_3}})),$$

$$C_{\theta} = M2^MC_{\gamma}^M \sum_{i=2}^{M} \sum_{n=1}^{N_i} \left\| (a_{i,n}) \right\|_{l=1}^{i-1},$$

$$\lambda = \min(1, \frac{\varepsilon^*}{4MC_{\gamma} \max(1, C_{\gamma}^M) \sum_{i=2}^{M} \sum_{n=1}^{N_i} \prod_{l=1}^{i} a_{i,n}}), \quad (2.2)$$

and

$$\beta = \min\left(1, \frac{1}{4} \cdot \frac{1 - \varepsilon^*}{2 \theta \|r_0\|_{\mathcal{H}^{m+2s}(0,T; U)}}, \frac{1}{4C_M \theta^2 \|r_0\|_{\mathcal{H}^{m+2s}(0,T; U)}} \right) \quad (2.3)$$

respectively. Here, $C_*$ is a constant given by lemma 13, $r_0$ is given by A.13.

**Definition 7** We say a sequence $\{u_n\}$ is Cauchy in a normed space $Y$ with respect to a norm $X$ if, for every $\varepsilon > 0$, there exists a $N$ such that for each $n_1, n_2 > N$,

$$\|u_{n_1} - u_{n_2}\|_X \leq \varepsilon$$

Let $v$ be a nonzero vector in $\mathbb{R}^N$, and for each $x \neq 0$ let $\angle(x, v)$ be the angle between the position vector $x$ and $v$. For given such $v, \rho > 0$, and $k$ satisfying $0 < k \leq \pi$, the set

$$C = \{x \in \mathbb{R}^N : x = 0 \text{ or } 0 < |x| \leq \rho, \angle(x, v) \leq k/2$$

is called a finite cone of height $\rho$, axis direction $v$ and aperture angle $k$ with vertex at the origin. Note that $x + C = \{x + y : y \in C\}$ is a finite cone with vertex at $x$ but the same dimensions and axis direction as $C$ and is obtained by parallel translation of $C$.

**Definition 8** $\Omega$ satisfies the cone condition if there exists a finite cone $C$ such that each $x \in \Omega$ is the vertex of a finite cone $C_x$ contained in $\Omega$ and congruent to $C$. Note that $C_x$ need not be obtained from $C$ by parallel translation, but simply by rigid motion.
Definition 9 Ω satisfies the strong local Lipschitz condition if there exists positive numbers δ and M, a locally finite open cover \{U_i\} of boundary Ω, and, for each j a real-value function \( f_j \) of \( n - 1 \) variables, such that the following conditions hold:

1. For some finite R, every collection of \( R + 1 \) of the sets \( U_i \) has empty intersection.
2. For every pair of points \( x, y \in Ω_δ \) such that \( |x - y| < δ \), there exists \( j \) such that
   \[ x, y \in V_j \equiv \{ x \in U_j : \text{dist}(x, \partial U_j) > δ \} \]
3. Each function \( f_j \) satisfies a Lipschitz condition with constant M: that is, if \( ξ = (ξ_1, ..., ξ_{n-1}) \) and \( ρ = (ρ_1, ..., ρ_{n-1}) \) are in \( ℝ^{n-1} \), then
   \[ |f(ξ) - f(ρ)| \leq M |ξ - ρ| \]
4. For some Cartesian coordinate system \( (ζ_{i,1}, ..., ζ_{i,n}) \) in \( U_j \), \( Ω \cap U_j \) is represented by the inequality
   \[ ζ_{i,n} < f_j(ζ_{i,1}, ..., ζ_{i,n}) \]

Definition 10 A Banach norm or complete norm is a norm that induces a complete metric. A normed space is a Banach space or B-space or complete normed space if its norm is a Banach norm.

2.2 The idea of construction

The main idea of this paper is to construct a Cauchy sequence \( \{u_k\} \) to approach the global solution of nonlinear parabolic equation (A.1) in the space \( W^{s+1.2}_{m+1}(0, T; U) \). The proposition 11 and 12 will show that \( W^{s+1.2}_{m+1}(0, T; U) \) is Banach space.

Assume that \( u_k \) is not the solution. It must have a nontrivial residue \( r_k \), where \( r_k \) is defined by
\[
r_k := (\frac{∂}{∂t} + L)u_k - f_0 - L_1 u_k.\tag{2.4}
\]

Let \( ρ_k s_k := u_{k+1} - u_k \), where \( ρ_k \) is a constant less than 1. Since \( f(u_{k+1}) \) is a polynomial of linear functional on \( u_{k+1} \), identity (A.1) can be decomposed into three parts;
\[
r_{k+1} = r_k + F_1(ρ_k s_k) + F_2(ρ_k s_k)\tag{2.5}
\]
where $F_1(\rho_k s_k)$ is given by

$$F_1(\rho s_k) := \left( \frac{\partial}{\partial t} + L \right)(\rho_k s_k) - \sum_{i=2}^{M} \sum_{n=1}^{N_i} \sum_{j=1}^{i} F_{i,j,n}(\rho_k s_k)(\prod_{l=1,l \neq j}^{i-1} F_{i,l,n}(u_k))$$

(2.6)

and $F_2(\rho_k s_k)$ is denoted by

$$F_2(\rho_k s_k) := r_{k+1} - r_k - F_1(\rho_k s_k)$$

(2.7)

$$= - \sum_{i=2}^{M} \sum_{j=2}^{i} \binom{i}{j} \prod_{l=1}^{j} F_{i,l,n}(\rho_k s_k)(\prod_{l=1,l \neq j}^{i-1} F_{i,l,n}(u_k))$$

where $j_l$ and $j_t$ form a partition of set $\{1, 2, \ldots, i\}$. Clearly, $F_1(\rho_k s_k)$ is a linear operator of $\rho s_k$, more precisely,

$$F_1(\rho s_k) = \rho_k F_1(s_k)$$

And $F_2(\rho s_k)$ is a nonlinear operator of $\rho s_k$ but is homogenous with order larger than 1.

In order to achieve the monotonic convergence of the residue $r_k$, we want

$$|r_{k+1}|_{N_m+2s}(0,T;U) \leq (1 - \beta) |r_k|_{N_m+2s}(0,T;U)$$

(2.8)

for some fixed constant $\beta \in (0, 1)$. Since we want $\{u_k\}$ to be Cauchy, we attempt to control the step length, $\rho_k s_k$, by

$$\|\rho_k s_k\|_{N_m+2s+1}(0,T;U) \leq \rho_k \theta \|r_k\|_{N_m+2s}(0,T;U)$$

(2.9)

where $\theta$ is independent of $k$ and $0 < \rho_k \leq 1$.

However, there may not be an $s_k$ such that

$$F_1(s_k) = -r_k$$

(2.10)

The idea here is to seek a perturbation $R_k$ of $r_k$ such that

$$\|R_k - r_k\|_{N_m+2s}(0,T;U) \leq \varepsilon^* \|r_k\|_{N_m+2s}(0,T;U)$$

(2.11)

for some small $\varepsilon^*$, and there exists an $s_k$ such that

$$F_1(s_k) = -R_k.$$
The method to construct such a $R_k$ is proposed by theorem 16. This theorem proves the existence of $R_k$ and $s_k$ and the regularity of the solution $\{s_k\}$.

In conclusion, we perturb $r_k$ such that equation (A.21) has a solution satisfying condition (A.17) and (A.18). The inequality (A.17) and (A.18) will guarantee the convergence of the Cauchy sequence $\{u_k\}$ in $\mathbb{R}_{m+2a+1}^{1,2}(0,T;U)$ for any $T$.

Based on these considerations, we propose the following algorithm:

Given the Navier-Stokes equations $f : \mathbb{R}^N \rightarrow \mathbb{R}^N$ continuously differentiable, and $\{u^0, p^0\}$ satisfies the above conditions. Let $u^0, p^0, s \geq 2, \infty > m > N/2 + 2 \geq 3$, and $\varepsilon^* \in (1, 1/2)$ be given. Let $\theta$ and $C^*$ be constant. Set

$$
\beta = \min\left(1, \frac{(1 - \varepsilon^*)^2}{2C^*\theta^2\|r^0_i\|_{W^{s,\infty}(0,T;H^m(U))}}\right)
$$

For step $k = 1$ until $\infty$ or stop, do

1. compute

$$
r^k_i := -f(x^k_i, p^k) \tag{2.13}
$$

and decide whether to stop or continuous. The function $f(x^k_i, p^k)$ is given by

$$
f(u^k_i, p^k) := \frac{\partial u^k_i}{\partial t} + \sum_j u^k_j \frac{\partial u^k_i}{\partial x_j} + \frac{\partial p^k}{\partial x_i} - \nu \Delta u^k_i
$$

2. Construct correction function $R^k_i$, subject to

$$
\left| R^k_i - r^k_i \right|_{W^{s,\infty}(0,T;H^m(U))} \leq \varepsilon^* \left| r^k_i \right|_{W^{s,\infty}(0,T;H^m(U))}
$$

and define

$$
h^k_i := R^k_i - r^k_i \tag{2.14}
$$

3. Solve $F_1(s^k_i, s^k_{N+1}) = R^k_i$ for $s^k_i, s^k_{N+1}$, or determine an approximation to them.

4. Choose a suitable constant $\rho^k$ such that

$$
\rho^k = \min(1.0, \frac{1}{2(1 - \varepsilon^*)} \frac{1 - \varepsilon^*}{2C^*\theta^2\|r^0_i\|_{W^{s,\infty}(0,T;H^m(U))}})
$$

5. Apply a scaled Newton correction, $u^{k+1}_i = u^k_i + \rho^k s^k_i$ and $p^{k+1} = p^k + \rho^k s^k_{N+1}$

6. Check for convergence and return to step 1 as necessary.
2.3 The cited lemmas and theorems

In this section, we list the lemmas and theorems cited in this thesis.

**Theorem 1** (Uniform convergence and differentiation, theorem 7.17 of Rudin (1976)) Suppose \( \{f_n\} \) is a sequence of functions, differentiable on \([a, b]\) and such that \( \{f_n(x_0)\} \) converges for some point \( x_0 \) on \([a, b]\). If \( \{f'_n\} \) converges uniformly on \([a, b]\), then \( \{f_n\} \) converges uniformly on \([a, b]\), to a function \( f \), and

\[
f'(x) = \lim_{n \to \infty} f'_n \quad (a \leq x \leq b).
\]

**Theorem 2** (Theorem 4.39, Adams (2003)) Let \( \Omega \) be a domain in \( \mathbb{R}^N \) satisfying the cone condition. If \( mp > N \) or \( p = 1 \) and \( m \geq N \), then there exists a constant \( K^* \) depending on \( m, p, N \), and the cone \( C \) determining the cone condition for \( \Omega \), such that for \( u, v \in W^{m, p}(\Omega) \) the product \( uv \), defined pointwise a.e. in \( \Omega \), satisfies

\[
\|uv\|_{W^{m, p}(\Omega)} \leq K^* \|u\|_{W^{m, p}(\Omega)} \|v\|_{W^{m, p}(\Omega)}
\]

In particular, equipped with the equivalent norm \( \|\cdot\|_{W^{m, p}(\Omega)}^* \) defined by

\[
\|u\|_{W^{m, p}(\Omega)}^* = K^* \|u\|_{W^{m, p}(\Omega)}.
\]

\( W^{m, p}(\Omega) \) is a commutative Banach algebra with respect to pointwise multiplication in that

\[
\|uv\|_{W^{m, p}(\Omega)}^* \leq \|u\|_{W^{m, p}(\Omega)}^* \|v\|_{W^{m, p}(\Omega)}^*
\]

**Theorem 3** (Sobolev imbedding theorem, 4.12 of Adams (2003)) Let \( \Omega \) be a domain in \( \mathbb{R}^N \) satisfying the cone condition. Let \( \Omega_k \) be the intersection of \( \Omega \) with a plane of dimension \( k \) in \( \mathbb{R}^N \). Let \( j \geq 0 \) and \( m \geq 1 \) be integers and let \( 1 \leq p < \infty \).

**Part 1** Suppose \( \Omega \) satisfies the cone condition.

**Case A** If either \( mp > N \) or \( m = n \) and \( p = 1 \), then

\( W^{j+m, p}(\Omega) \to C^j_B(\Omega) \).

Moreover, if \( 1 \leq k \leq N \), then

\( W^{j+m, p}(\Omega) \to W^{j, q}(\Omega_k) \quad \text{for} \ p \leq q \leq \infty \)
and, in particular
\[ W^{m,p}(\Omega) \to L^q(\Omega_k) \quad \text{for} \ p \leq q \leq \infty \]

**Case B** If \( 1 \leq k \leq N \) and \( mp = N \), then
\[ W^{j+m,p}(\Omega) \to W^{j,q}(\Omega_k) \quad \text{for} \ p \leq q < \infty \]

and, in particular
\[ W^{m,p}(\Omega) \to L^q(\Omega_k) \quad \text{for} \ p \leq q < \infty \]

**Case C** If \( mp < N \) and either \( N - mp < k \leq N \) or \( p = 1 \) and \( n - m = k \leq N \), then
\[ W^{j+m,p}(\Omega) \to W^{j,q}(\Omega_k) \quad \text{for} \ p \leq q \leq p^* = kp/(N - mp) \]

In particular,
\[ W^{m,p}(\Omega) \to L^q(\Omega_k) \quad \text{for} \ p \leq q \leq p^* = kp/(N - mp). \]

The imbedding constants for the imbedding above depend only on \( m, p, q, j, k \), and the dimensions of the cone \( C \) in the cone condition.

**Part II** Suppose \( \Omega \) satisfies the strong local Lipschitz condition. Then the target space \( C^{j}_R(\Omega) \) in the imbedding (1) can be replaced with the smaller space \( C^{j}(\Omega) \), and the imbedding can be further refined as follows:

If \( mp > N > (m - 1)p \), then
\[ W^{j+m,p}(\Omega) \to C^{j,\lambda}(\Omega) \quad \text{for} \ 0 < \lambda \leq m - (N/p). \]

If \( N = (m - 1)p \), then
\[ W^{j+m,p}(\Omega) \to C^{j,\lambda}(\Omega) \quad \text{for} \ 0 < \lambda < 1. \quad (2.16) \]

Also, if \( N = m - 1 \) and \( p = 1 \), then (2.16) holds for \( \lambda = 1 \) as well.

**Part III** All of the imbedding in Part I and II are valid for arbitrary domains \( \Omega \) if the \( W - \)space undergoing the imbedding is replaced with the corresponding \( W_0 - \)space.
**Theorem 4** (Theorem 8.37, Gilbarg and Trudinger (2001)) Let $L$ be a self-adjoint operator such that $L$ is strictly elliptic and has bounded coefficients. Then $L$ has a countably infinite discrete set of eigenvalues, $\Sigma = \{\sigma_m\}$, given by

$$\sigma_m = \inf \{J(u) \mid u \neq 0, \ (u,v) = 0 \ \forall v \in \{V_1, \ldots, V_{m-1}\}$$

whose eigenfunctions span $H$.

**Theorem 5** (Theorem 9.19, Gilbarg and Trudinger (2001)) Let $u$ be a $W^{2,p}_{loc}(\Omega)$ solution of the elliptic equation $Lu = f$ in a domain $\Omega$, where the coefficients of $L$ belong to $C^{k-1,1}(\Omega)$, $(C^{k-1,1}(\Omega))$, $f \in W^{k,p}_{loc}(\Omega)$, $(C^{k-1,1}(\Omega))$, with $1 < p, q < \infty$, $k \geq 1$, $0 < \alpha < 1$. Then $u \in W^{k+2,p}_{loc}(\Omega)$, $(C^{k+1,1}(\Omega))$. Furthermore, if $\Omega \in C^{k+1,1}(\Omega)$, $(C^{k+1,1}(\Omega))$, $L$ is strictly elliptic in $\Omega$ with coefficients in $C^{k-1,1}(\Omega)$, $(C^{k-1,1}(\Omega))$, and $f \in W^{k,p}(\Omega)$, $(C^{k-1,1}(\Omega))$, then $u \in W^{k+2,p}(\Omega)$, $(C^{k+1,1}(\Omega))$. 

CHAPTER 3. MAIN THEOREMS

We will show that the space $\mathfrak{S}^s(0, T; H^m(U))$ and $\mathfrak{K}^q_m(0, T; U)$ are the Banach space for any finite $s, m \geq 0$ and $q \geq 2$.

**Theorem 6** The space $\mathfrak{S}^s(0, T; H^m(U))$ is a Banach space if $U \subset \mathbb{R}^N$ satisfies the cone condition and $m > N/2$.

**Proof.** (1) Clearly, $\mathfrak{S}^s(0, T; H^m(U))$ is a norm space. We will show next that it is complete. Suppose that $\{u_n\}$ is Cauchy in $\mathfrak{S}^s(0, T; H^m(U))$. For every given $t$, $\{\frac{\partial^j u_n}{\partial t^j}\}$ is Cauchy in $H^m(U)$, where $0 \leq j \leq s$, because for any pair of $u_{n_1}$ and $u_{n_2}$, we have

$$\left\| \frac{\partial^j u_{n_1}}{\partial t^j} - \frac{\partial^j u_{n_2}}{\partial t^j} \right\|_{H^m(U)} (t) \leq \max_{t \in [0, T]} \left\| \frac{\partial^j u_{n_1}}{\partial t^j} - \frac{\partial^j u_{n_2}}{\partial t^j} \right\|_{H^m(U)}$$

$$\leq \left\| u_{n_1} - u_{n_2} \right\|_{\mathfrak{S}^s(0, T; H^m(U))}.$$

Considering that $H^m(U)$ is closed, for any given $t$, $\frac{\partial^j}{\partial t^j} u_n$ converges to function $v_j(t, x)$ in $\mathfrak{S}^0(0, T; H^m(U))$. It is easy to show that $v_j(t, x)$ is well defined and $v_j \in \mathfrak{S}^0(0, T; H^m(U))$. By the imbedding theorem, $H^m(U) \rightarrow C^r_B(U)$ where $r$ is the largest integer less than $m - N/2$. i.e., there exist a constant $k$ such that

$$\left\| \frac{\partial^j u_{n_1}}{\partial t^j} - \frac{\partial^j u_{n_2}}{\partial t^j} \right\|_{C^r_B(U)} (t) \leq k \left\| \frac{\partial^j u_{n_1}}{\partial t^j} - \frac{\partial^j u_{n_2}}{\partial t^j} \right\|_{H^m(U)} (t)$$

$$\leq k \left\| u_{n_1} - u_{n_2} \right\|_{\mathfrak{S}^s(0, T; H^m(U))}$$

(3.1)

where $k$ is independent of $j, n$, and $t$. Thus, $\{u_n\}$ uniformly converge in $\mathfrak{S}^0(0, T; C^r_B(U))$ and $u_n, v_j \in \mathfrak{S}^0(0, T; C^r_B(U))$ for any $0 \leq j \leq s$.

(2) In this step, we will show that $v_j(t, x) = \frac{\partial^{j-1} u_n}{\partial t^{j-1}}$ in the space $\mathfrak{S}^0(0, T; C^r_B(U))$ for any $0 < j \leq s$. At first, we will prove that $\frac{\partial^{j-1} u_n}{\partial t^{j-1}}$ and $\frac{\partial^j u_n}{\partial t^j}$ are uniformly converge to $v_j(t, x)$
and \( v_j(t, x) \) in \([0,T] \times U\). By the definition of norm,

\[
\max_{|\alpha| \leq j} \sup_{x \in U} \left| \frac{\partial^j u_{n_1}}{\partial t^j} (t, x) - \frac{\partial^j u_{n_2}}{\partial t^j} (t, x) \right| = \left\| \frac{\partial^j u_{n_1}}{\partial t^j} - \frac{\partial^j u_{n_2}}{\partial t^j} \right\|_{\mathcal{C}_F^r(U)} (t) \\
\leq k \left\| u_{n_1} - u_{n_2} \right\|_{\mathcal{C}^r(0,T; H^m(U))}
\]

Considering that sequence \( \{u_n\} \) is Cauchy, we have

\[
\lim_n \left| \frac{\partial^j u_n}{\partial t^j} - v_j \right| (t, x) \leq k \left\| u_{n_1} - v_j \right\|_{\mathcal{C}^r(0,T; H^m(U))} \tag{3.2}
\]

for every \( x \) and \( t \). This proves that the sequence \( \{\frac{\partial^{j-1} u_n}{\partial t^{j-1}}\} \) and \( \{\frac{\partial^j u_n}{\partial t^j}\} \) converge to \( v_{j-1}(t, x) \) and \( v_j(t, x) \) uniformly in \([0,T] \times U\).

For every given \( x \) in \( U \), by the uniform convergence and differentiation theorem (theorem 7.17, Rudin (1976)), we have \( \frac{\partial}{\partial t} v_{j-1}(t, x) = v_j(t, x) \) for this given \( x \). Since \( x \) is arbitrary, we conclude that \( \frac{\partial}{\partial t} v_{j-1}(t, x) = v_j(t, x) \) in \([0,T] \times U\). Moreover, we have \( \frac{\partial}{\partial t} v_{j-1}(t, x) = v_j(t, x) \) in \( \mathcal{D}^0(0,T; H^m(U)) \).

(3) Now, we are going to show that \( v_j(t, x) = \frac{\partial^{j-1} u}{\partial t^{j-1}} \) in the space \( \mathcal{D}^0(0,T; H^m(U)) \) for any \( 0 < j \leq s \). By the idea of induction, we assume \( v_j(t, x) = \frac{\partial^{j-1} u}{\partial t^{j-1}} \) in the space \( \mathcal{D}^0(0,T; H^m(U)) \) where \( 0 < j \leq s \) and \( r < m \). Let \( \Phi \) be a test function, thus

\[
\left( \frac{\partial^{j-1} u}{\partial t^{j-1}} - v_j(t, x), D^{r+1} \Phi \right) = \lim_n \left( \frac{\partial}{\partial t} \frac{\partial^{j-1} u_n}{\partial t^{j-1}} - \frac{\partial^j u_n}{\partial t^j}, D^{r+1} \Phi \right) \\
= \lim_n (-1)^{r+1} \left( D^{r+1} \frac{\partial}{\partial t} \frac{\partial^{j-1} u_n}{\partial t^{j-1}} - D^{r+1} \frac{\partial^j u_n}{\partial t^j}, \Phi \right) \\
= 0.
\]

Considering that \( j \) and \( r \) are arbitrary, we complete the proof. \( \blacksquare \)

We know that \( C^s([0,T]) \) is not Banach space, so does \( W^{s,\infty}([0,T]) \). However, the combination of \( C^s([0,T]) \) and \( H^m(U) \), \( C^s([0,T]) \times H^m(U) \), is a Banach space if the boundary of \( U \) satisfies the cone condition and \( m > N/2 \). This is not clear whether the theorem 11 can be satisfied for more general domain \( U \) or not.

**Theorem 7** The space \( \mathcal{C}^s_m(0,T; U) \) is a Banach space if \( U \subset \mathbb{R}^N \) satisfies the cone condition and \( m > N/2 \).
Proof. Suppose sequence \( \{u_n\} \) is Cauchy in \( \mathcal{N}_m^s(0,T;U) \). By theorem 11, sequence \( \{u_n\} \) converges to \( v_{s-j} \) in \( \mathcal{S}^{s-j}(0,T;H^{m+qj}(U)) \) for each \( 0 \leq j \leq s \). To prove the proposition, it suffices to show that \( v_{s-j} \in \mathcal{S}^{s-j-1}(0,T;H^{m+qj+q}(U)) \) for each \( 0 \leq j \leq s \). Since \( \mathcal{S}^{s-j}(0,T;H^{m+qj}(U)) \subset \mathcal{S}^{s-j-1}(0,T;H^{m+qj+q}(U)) \), we obtain

\[
v_{s-j} \in \mathcal{S}^{s-j-1}(0,T;H^{m+qj}(U))
\]

Thus, for any \( t \) and \( k \), where \( 0 \leq k \leq s - j - 1 \), \( \frac{\partial^{k} u_n}{\partial t^k} \) converges to \( \frac{\partial^{k} v_{s-j}}{\partial t^k} \) in \( H^{m+qj}(U) \) uniformly. Since \( \{u_n\} \) is Cauchy in \( \mathcal{N}_m^s(0,T;U) \) and \( q \geq 1 \), sequence \( \{ \frac{\partial}{\partial t} \frac{\partial^{k} u_n}{\partial t^k} \} \) converge in \( H^{m+qj}(U) \) uniformly. For any \( 0 \leq k \leq s - j - 1 \) and \( 0 \leq r \leq m + qj + q \),

\[
\left( \frac{\partial^{k} v_{s-j}}{\partial t^k}, D^r \Phi \right) = \lim_n \left( \frac{\partial^{k} u_n}{\partial t^k}, D^r \Phi \right) = \lim_n \left( (-1)^r \left( D^r \frac{\partial^{k} u_n}{\partial t^k}, \Phi \right) \right).
\]

We know that \( \lim_n D^r \frac{\partial^{k} u_n}{\partial t^k} \) exists in \( H^{m+qj+q} \) for each \( t \) and \( k \). Thus,

\[
v_{s-j} \in \mathcal{S}^{s-j-1}(0,T;H^{m+qj+q}(U))
\]

This completes the proof. ■

As a prelude to existence considerations we derive now some lemmas.

**Lemma 8** Let \( u_1, u_2 \in \mathcal{S}^s(0,T;W^{m,p}(U)) \) and \( u_3, u_4 \in \mathcal{N}_m^s(0,T;U) \). Let \( U \) be a set in \( \mathbb{R}^N \) satisfying a cone property. For each \( \infty > pm > N, q > 1, \) and \( \infty > s \geq 0, \) there exists a constant \( C_* = C_*(N,U,m,p,s) \) such that

\[
\| u_1 u_2 \|_{3^s(0,T;W^{m,p}(U))} \leq C_* \| u_1 \|_{3^s(0,T;W^{m,p}(U))} \| u_2 \|_{3^s(0,T;W^{m,p}(U))},
\]

\[
\| u_1 u_2 \|_{W^{s,\infty}(0,T;W^{m,p}(U))} \leq C_* \| u_1 \|_{W^{s,\infty}(0,T;W^{m,p}(U))} \| u_2 \|_{W^{s,\infty}(0,T;W^{m,p}(U))},
\]

and

\[
\| u_3 u_4 \|_{\mathcal{N}_m^s(0,T;U)} \leq C_* \| u_3 \|_{\mathcal{N}_m^s(0,T;U)} \| u_4 \|_{\mathcal{N}_m^s(0,T;U)}.
\]
Proof. Suppose \( u_1, u_2 \in \mathfrak{S}^s(0,T;W^{m,p}(U)) \). Let \( \gamma \) be such that \( s \geq \gamma \geq 0 \), then \( \frac{\partial^{\gamma} u_1}{\partial t^\gamma}, \frac{\partial^{\gamma} u_2}{\partial t^\gamma} \in W^{m,p}(U) \). Note that \( U \) holds a cone property and \( \infty > pm > N \). Due to the theorem 4.39 of Adams (2003), there exists a constant \( c_1 \) such that for any \( pm > N \),

\[
\left\| \frac{\partial^{\gamma} u_1}{\partial t^\gamma}, \frac{\partial^{\gamma} u_2}{\partial t^\gamma} \right\|_{W^{m,p}(U)} \leq c_1(U, N, m, p) \left\| \frac{\partial^{\gamma} u_1}{\partial t^\gamma} \right\|_{W^{m,p}(U)} \left\| \frac{\partial^{\gamma} u_2}{\partial t^\gamma} \right\|_{W^{m,p}(U)}
\]

where \( c_1 \) only depends \( U, N, m \) and \( p \).

The boundedness of \( u_1u_2 \) in \( \mathfrak{S}^s(0,T;W^{m,p}(U)) \) follows:

\[
\left\| u_1u_2 \right\|_{\mathfrak{S}^s(0,T;W^{m,p}(U))} = \max_{0 \leq t \leq T} \left\| \frac{\partial^{\gamma} u_1}{\partial t^\gamma} \right\|_{W^{m,p}(U)} \leq \max_{0 \leq t \leq T} \sum_{\gamma=0}^{s} \binom{s}{\gamma} c_1 \left\| \frac{\partial^{\gamma} u_1}{\partial t^\gamma} \right\|_{W^{m,p}(U)} \left\| \frac{\partial^{\gamma} u_2}{\partial t^\gamma} \right\|_{W^{m,p}(U)}
\]

Note that for each given \( t \in [0, T] \), \( \left\| \frac{\partial^{\gamma} u_1}{\partial t^\gamma} \right\|_{W^{m,p}(U)} \) and \( \left\| \frac{\partial^{\gamma} u_2}{\partial t^\gamma} \right\|_{W^{m,p}(U)} \) are in \( L^\infty([0,T]) \). Therefore, \( \| u_1u_2 \|_{\mathfrak{S}^s(0,T;W^{m,p}(U))} \) is bounded. Furthermore, there exists a constant \( C_* \), which is independent of \( T \), such that

\[
\left\| u_1u_2 \right\|_{\mathfrak{S}^s(0,T;W^{m,p}(U))} \leq C_* \left( \| u_1 \|_{\mathfrak{S}^s(0,T;W^{m,p}(U))} \| u_2 \|_{\mathfrak{S}^s(0,T;W^{m,p}(U))} \right).
\]

By repeating above deduction, we conclude that for each \( 0 \leq j \leq s \), there exists a constant \( c_j \) such that

\[
\| u_3u_4 \|_{\mathfrak{S}^{s-j,0}(0,T;W^{m,q,j,p}(U))} \leq c_j \| u_3 \|_{\mathfrak{S}^{s-j,0}(0,T;W^{m,q,j,p}(U))} \| u_4 \|_{\mathfrak{S}^{s-j,0}(0,T;W^{m,q,j,p}(U))}.
\]

Let \( C_* = \max(c_j, c, C'_*) \), we obtain:

\[
\| u_3u_4 \|_{\mathfrak{S}^{s,q}(0,T;U)} \leq C_* \| u_3 \|_{\mathfrak{S}^{s,q}(0,T;U)} \| u_4 \|_{\mathfrak{S}^{s,q}(0,T;U)}
\]

and

\[
\left\| u_1u_2 \right\|_{\mathfrak{S}^s(0,T;W^{m,p}(U))} \leq C_* \left( \| u_1 \|_{\mathfrak{S}^s(0,T;W^{m,p}(U))} \| u_2 \|_{\mathfrak{S}^s(0,T;W^{m,p}(U))} \right).
\]

In the same way, we can prove the inequality (A.25).

Preceding lemma enables us to bound the norm of a polynomial by the corresponding polynomial of norm. With these inequalities, we can easily prove that the nonlinear operator \( F_2 \) is bounded by \( r_k \).
Remark 1 Suppose \( \varphi_i \) to be the normalized eigenfunctions of the Dirichlet problem for the operator \( Lu \) denoted by (A.4). The solution of generated Dirichlet problem of the heat equation

\[
 u_t + Lu = f, \\
 u(0, U) = g(U), \quad u(t, \partial U) = h(t; \partial U),
\]

is given by

\[
 u = \sum_{i=1}^{\infty} e^{-\lambda_i t} \varphi_i \left( \int_U g_t \varphi_i + \int_0^t e^{\lambda_i \tau} \int_U \varphi_i (\xi) f(\tau; \xi) d\xi d\tau - \int_0^t \int_{\partial U} h \frac{\partial \varphi_i (\xi)}{\partial \xi_n} d\xi d\tau \right).
\]

Denote

\[
 \psi(t; x|\tau; \xi) = \sum_{i=1}^{\infty} e^{-\lambda_i (t-\tau)} \varphi_i (x) \varphi_i (\xi), 
\]

(3.6)

It yields

\[
 u = \int_0^t \int_U \psi(t; x|\tau; \xi) f(\tau; \xi) d\xi d\tau + \int_0^t \int_U \psi(t; x|0; \xi) g(\xi) d\xi d\tau - \int_0^t \int_{\partial U} h \frac{\partial \psi}{\partial \xi_n}.
\]

Here, \( \psi(t; x|\tau; \xi) \) is the Dirichlet heat kernel. The regularity of \( \psi(t; x|\tau; \xi) \) depends on the smoothness of the domain \( U \) and the coefficients of operator \( L \).

Lemma 9 Under condition (A1) and (A2), the operator \( Lu \) has a set of eigenfunctions \( \{ \varphi_{\lambda_i} \} \) obeying (1) \( \{ \varphi_{\lambda_i} \} \) form an orthonormal basis of a Hilbert space \( H(U) \), (2) \( \varphi_{\lambda_i} \in H^3_0(U) \) and the eigenvalues \( \lambda_i \) hold

\[
 0 < \lambda_1 < \ldots < \lambda_n < \ldots, 
\]

(3.7)

and (3) \( \forall 0 \leq \beta \leq m + 2(1 + s) \) and \( \varphi_i, \varphi_j \in \{ \varphi_{\lambda_i} \} \):

\[
 \left\langle D^\beta \varphi_i, D^\beta \varphi_j \right\rangle = 0, \quad \text{if } i \neq j \quad (3.8)
\]

\[
 \left\langle D^\beta \varphi_i, D^\beta \varphi_i \right\rangle \leq \frac{\lambda_i - \gamma_3}{\gamma_1} \left\langle D^{\beta-1} \varphi_i, D^{\beta-1} \varphi_i \right\rangle \quad (3.9)
\]

where \( \gamma_1 \) is given by condition (A2).

Proof. It is known that the eigenfunctions \( \varphi_{\lambda_i}^1 \in H^{m+2(s+1)}(U) \cap H^1_0(U) \) of operator \( Lu \) span a Hilbert space \( H(U) \), and the corresponding eigenvalues hold (A.28) (Theorem 8.37 and 9.19,
Denote $\varphi_{\lambda_i} = \lim_{\varepsilon \to 0} \eta_\varepsilon \otimes \varphi_{\lambda_i}^2$, where $\eta_\varepsilon$ is a standard mollifier and

$$
\varphi_{\lambda_i}^2 = \varphi_{\lambda_i}^1, \quad \text{when } x \in U,
$$
$$
\varphi_{\lambda_i}^3 = 0, \quad \text{when } x \in \mathbb{R}^N \setminus U.
$$

Since $U$ is a bounded open set in $\mathbb{R}^N$, $\varphi_{\lambda_i} \in H_0^{m+2(s+1)}(\overline{U})$ and $\varphi_{\lambda_i} = \varphi_{\lambda_i}^1$ in $H^{m+2(s+1)}(U) \cap H_0^1(U)$. Then, $\{\varphi_{\lambda_i}\}$ span $H(U)$. Meanwhile, eigenvalues of $\varphi_{\lambda_i}$ is equivalent to that of $\varphi_{\lambda_i}^1$. Thus, (A.28) is satisfied. The orthogonality of $\{\varphi_{\lambda_i}\}$ in $H(U)$ immediately follows the orthogonality of $\{\varphi_{\lambda_i}^1\}$ in $H(U)$. The reminded problem is to show the orthogonality of $\{D^\beta \varphi_{\lambda_i}\}$ for any $0 < \beta \leq m + 2(1 + s)$. Suppose that $\varphi_i, \varphi_j$ are arbitrary in $\{\varphi_{\lambda_i}\}$ and $i \neq j$. The Green’s first identity Gilbarg and Trudinger (2001) and the facts $\varphi_i, \varphi_j \in H_0^{m+2(s+1)}(\overline{U})$ immediately follow

$$
\lambda_j \left\langle D^{\beta-1} \varphi_i, D^{\beta-1} \varphi_j \right\rangle = \left\langle D^{\beta-1} \varphi_i, LD^{\beta-1} \varphi_j \right\rangle = \left\langle \alpha^{ij} D^{\beta} \varphi_i, D^{\beta} \varphi_j \right\rangle + \left\langle b D^{\beta-1} \varphi_i, D^{\beta-1} \varphi_j \right\rangle
$$

where operator $L$ is defined by the condition (A2). Together with the strong elliptic condition (A.6)-(A.8), we obtain:

$$
\lambda_j \left\langle D^{\beta-1} \varphi_i, D^{\beta-1} \varphi_j \right\rangle \geq \gamma_1 \left\langle D^{\beta} \varphi_i, D^{\beta} \varphi_j \right\rangle + \gamma_3 \left\langle D^{\beta-1} \varphi_i, D^{\beta-1} \varphi_j \right\rangle.
$$

It is clear that $\lambda_i > \gamma_3$ because

$$
\lambda_j \geq \min_{\|u\|=1, u \in H_0^1(U)} \left\langle u, Lu \right\rangle \geq \min_{\|u\|=1, u \in H_0^1(U)} \gamma_1 \|Du\| + \gamma_3. \tag{3.10}
$$

Thus, $\left\langle D^{\beta} \varphi_i, D^{\beta} \varphi_j \right\rangle \leq \frac{\lambda_i - \gamma_3}{\gamma_1} \left\langle D^{\beta-1} \varphi_i, D^{\beta-1} \varphi_j \right\rangle = \left(\frac{\lambda_i - \gamma_3}{\gamma_1}\right)^\beta \left\langle \varphi_i, \varphi_j \right\rangle$. This proves the inequality (A.30).

The following theorem deals with the upper bound of the heat kernel.

**Theorem 10** Denote $F(g) := \int_0^t \int_{\mathbb{R}^d} \psi(0; x|\tau; y)g(t - \tau; y)d\tau d\xi$ and let $X_1$ to be the space $W^{1,2}(0, T; H^{r-1}(U)) \cap L^2(0, T; H^{r+1}(U)) \cap L^\infty(0, H^r(U))$. Under condition (A1) and (A2), for any $\gamma \geq 1$, if $g \in X_1$, then
(1) there exist follow inequalities;

\[ \|F(g)\|_{L^2(0,T;H^{\gamma}(U))} \leq C_1 \|g_0\|_{W^{1,2}(0,t;H^{\gamma-1}(U))} + C_2 \|g_0(0; x)\|_{H^\gamma(U)} \]  \hspace{1cm} (3.11)

and

\[ \|F(g)\|_{L^2(0,T;H^{\gamma+1}(U))} \leq C_1 \|g_0\|_{L^2(0,t;H^{\gamma+1}(U))} \]  \hspace{1cm} (3.12)

where \( C_1 = N^{\gamma+1} \max(\frac{1}{\sqrt{\gamma}}, \min(\sqrt{T}, \frac{1}{\sqrt{\gamma}})) \) and \( C_2 = N^{\gamma+1} \).

(2) \( F(g) \) is a compact linear operator from \( X_1 \) into \( H_{\gamma,2}^{1,2}(0, T; U) \).

**Proof.** (1) Firstly, we estimate the upper bound of \( \|F(g)\|_{L^2(0,T;H^{\gamma}(U))} \). It suffices to show that, for any \( 0 \leq \beta \leq \gamma \), \( \|D^3 \frac{\partial}{\partial t} F(g_0)\|_{L^2(0,T;H^{\gamma}(U))} \) is bounded by \( \|g_0\|_{W^{1,2}(0,t;H^{\gamma-1}(U))} \) and \( \|g_0\|_{L^2(0,t;H^{\gamma+1}(U))} \). According to the Leibnitz integral rule, we compute

\[ \left\| \frac{D^3}{\partial t} F(g_0) \right\|_{L^2(U)} \leq \left\{ \int_U dx \left( \int_0^t \int_U \left( \int_0^t \frac{\partial}{\partial t} \int_U D^3 \psi(0; x|\tau; y) g_0(t - \tau; y) dy \right)^2 \right)^{1/2} \leq \left\{ \int_U dx \left[ \int_0^t \int_U \int_U D^3 \psi(0; x|\tau; y) \frac{\partial}{\partial t} g_0(t - \tau; y) dy \right]^2 \right]^{1/2} + \left\{ \int_U dx \left[ \int_U D^3 \psi(0; x|t; y) g_0(t; y) dy \right]^2 \right]^{1/2} . \]

It follows from the identity (A.27),

\[ \left\| \frac{D^3}{\partial t} F(g_0) \right\|_{L^2(U)} \leq \left\{ \int_U dx \left[ \int_0^t \sum_{i=1}^{\infty} e^{-\lambda_i \tau} D^3 \phi_i(x) \left( \phi_i(\xi), \frac{\partial g_0}{\partial t} \right) \right]^2 \right\}^{1/2} + \left\{ \int_U \left[ \sum_{i=1}^{\infty} D^3 \phi_i(x) e^{-\lambda_i \tau} \left( \phi_i(\xi), g_0(0; y) \right) \right]^2 \right\}^{1/2} . \]

Without loss of generality, we discuss the case \( \beta \geq 1 \) firstly. Due to the lemma 14, \( D^3 \phi_i(x) \) are orthogonal. Thus,

\[ I = \left\{ \sum_{i=1}^{\infty} \left\| D^3 \phi_i(x) \right\|_{L^2(U)}^2 \left[ \int_0^t e^{-\lambda_i \tau} \left( \phi_i(\xi), \frac{\partial g_0}{\partial t} \right) \right] \right\}^{1/2} . \]

The inequality (A.30) follows that

\[ I \leq \left\{ \sum_{i=1}^{\infty} \frac{(\lambda_i - \gamma_3)}{2 \gamma_i \lambda_i} \left\| D^{\beta-1} \phi_i(x) \right\|_{L^2(U)}^2 \left[ \int_0^t \sqrt{2 \lambda_i} e^{-\lambda_i \tau} \left( \phi_i(\xi), \frac{\partial g_0}{\partial t} \right) \right]^2 \right\}^{1/2} . \]
Clearly, \( \frac{(\lambda_i - \gamma_3)}{2\gamma_1} \leq \frac{1}{2\gamma_1} \). By the Hölder’s inequality,

\[
I \leq \frac{1}{\sqrt{2\gamma_1}} \left( \sum_{i=1}^{\infty} \| D^{\beta-1} \varphi_i(x) \|_{L^2(U)}^2 \right) \int_0^t 2\lambda_i e^{-2\lambda_i \tau} \int_0^t \left( \varphi_i(\xi), \frac{\partial g_i}{\partial t} \right)^2 \right) \frac{1}{2}
\]

\[
\leq \frac{1}{\sqrt{2\gamma_1}} \left( \int_0^t \sum_{i=1}^{\infty} \| D^{\beta-1} \varphi_i(x) \|_{L^2(U)}^2 \left( \varphi_i(\xi), \frac{\partial g_i}{\partial t} \right)^2 \right) \frac{1}{2}
\]

\[
= \frac{1}{\sqrt{2\gamma_1}} \left( \sqrt{2\gamma_1} \left\| D^{\beta-1} \frac{\partial g_i}{\partial t} \right\|_{L^2(0,T;L^2(U))} \right)
\]

because \( D^{\beta-1} \varphi_i(x) \) are orthogonal.

In the same way for \( I \), we have

\[
II = \sum_{i=1}^{\infty} \left\| D^\beta \varphi_i(x) \right\|_{L^2(U)}^2 \left( \varphi_i(\xi), \frac{\partial g_i}{\partial t} \right)^2 \frac{1}{2}
\]

\[
\leq \left\| D^\beta g_i(0; y) \right\|_{L^2(U)}.
\]

because \( 0 < e^{-2\lambda_i \tau} \leq 1 \) and \( D^\beta \varphi_i(x) \) are orthogonal. Thus, for any \( t \in [0, T] \),

\[
\left\| \frac{\partial}{\partial t} F(g_i) \right\|_{L^2(U)} \leq \frac{1}{\sqrt{2\gamma_1}} \left\| D^{\beta-1} \frac{\partial g_i}{\partial t} \right\|_{L^2(0,T;L^2(U))} + \left\| D^\beta g_i(0; y) \right\|_{L^2(U)}
\]

For the case \( |\beta| = 0 \), by the analogous method with above, we deduce

\[
\left\| \frac{\partial}{\partial t} F(g_i) \right\|_{L^2(U)} \leq \left\| \varphi_i \right\|_{L^2(U)} \left\| \int_0^t e^{-2\lambda_i \tau} \left( \varphi_i(\xi), \frac{\partial g_i}{\partial t} \right)^2 \right\|_{L^2(U)} + \left\| g_i(0; y) \right\|_{L^2(U)}.
\]

In the lemma 14, we prove that \( \lambda_i > \gamma_3 \). Hence, \( \int_0^T e^{-2\lambda_i \tau} \leq \min(\sqrt{T}, \frac{1}{\sqrt{2\gamma_3}}) \). Thereby,

\[
\left\| \frac{\partial}{\partial t} F(g_i) \right\|_{L^2(U)} \leq \min(\sqrt{T}, \frac{1}{\sqrt{2\gamma_3}}) \left\| \frac{\partial g_i}{\partial t} \right\|_{L^\infty(0,T;H^1(U))} + \left\| g_i(0; y) \right\|_{L^2(U)}.
\]

In summary,

\[
\left\| F(g) \right\|_{L^1(0,T;H^\gamma(U))} \leq \frac{1}{\sqrt{2\gamma_1}} \sum_{1 \leq |\beta| \leq \gamma} \left\| D^{\beta-1} \frac{\partial g_i}{\partial t} \right\|_{L^2(0,T;H^1(U))} + \sum_{1 \leq |\beta| \leq \gamma} \left\| D^\beta g_i(0; x) \right\|_{L^2(U)}
\]

\[
+ \min(\sqrt{T}, \frac{1}{\sqrt{2\gamma_3}}) \left\| \frac{\partial g_i}{\partial t} \right\|_{L^\infty(0,T;H^1(U))} + \left\| g_i(0; y) \right\|_{L^2(U)}
\]

\[
\leq \min(\sqrt{T}, \frac{1}{\sqrt{2\gamma_3}}) \left\| \frac{\partial g_i}{\partial t} \right\|_{L^\infty(0,T;H^1(U))} + \min(\sqrt{T}, \frac{1}{\sqrt{2\gamma_3}}) \left\| g_i(0; y) \right\|_{L^2(U)}
\]

\[
+ \left\| g_i(0; x) \right\|_{H^\gamma(U)} \sum_{|\beta| \leq \gamma} 1
\]

\[
\leq C_1 \left\| g_i \right\|_{L^1(0,T;H^\gamma(U))} + C_2 \left\| g_i(0; x) \right\|_{H^\gamma(U)}
\]
where \( C_1 = N^{\gamma+1} \max\left(\frac{1}{\sqrt{2\gamma}}, \min(\sqrt{T}, \frac{1}{\sqrt{2\gamma}})\right) \) and \( C_2 = N^{\gamma+1} \). This proves inequality (A.32).

(2) We will show the inequality (A.33) in this part. For the case \(|\beta| > 0\). In the same way for above, we have

\[
\|D^\beta F(g_t)\|_{L^2(U)} = \left\{ \int_U dx \left[ \int_0^t d\tau \sum_{i=1}^{\infty} e^{-\lambda_i \tau} D^\beta \varphi_i(x) \langle \varphi_i(x), g_t \rangle \right]^2 \right\}^{1/2}
\]

\[
\leq \left\{ \sum_{i=1}^{\infty} \frac{1}{2\lambda_i} \left\| D^\beta \varphi_i(x) \right\|_{L^2(U)}^2 \int_0^t 2\lambda_i e^{-2\lambda_i \tau} \int \langle \varphi_i(x), g_t \rangle^2 \right\}^{1/2}
\]

\[
\leq \frac{1}{\sqrt{2\gamma_1}} \left\{ \sum_{i=1}^{\infty} \left\| D^{\beta-1} \varphi_i(x) \right\|_{L^2(U)}^2 \int_0^t 2\lambda_i e^{-2\lambda_i \tau} \int \langle \varphi_i(x), g_t \rangle^2 \right\}^{1/2}
\]

\[
\leq \frac{1}{\sqrt{2\gamma_1}} \left\| D^{\beta-1} g_t \right\|_{L^2(0,t;L^2(U))}.
\]

For the case \( \beta = 0 \),

\[
\|F(g_t)\|_{L^2(U)} = \left\{ \int_U dx \left[ \int_0^t d\tau \sum_{i=1}^{\infty} e^{-\lambda_i \tau} \varphi_i(x) \langle \varphi_i(x), g_t \rangle \right]^2 \right\}^{1/2}
\]

\[
\leq \min(\sqrt{T}, \frac{1}{\sqrt{2\gamma_3}}) \left\{ \sum_{i=1}^{\infty} \left\| \varphi_i(x) \right\|_{L^2(U)}^2 \int_0^t \langle \varphi_i(x), g_t \rangle^2 \right\}^{1/2}
\]

\[
= \min(\sqrt{T}, \frac{1}{\sqrt{2\gamma_3}}) \left\| g_t \right\|_{L^2(0,t;L^2(U))}.
\]

Thus,

\[
\|F(g_t)\|_{L^2(0,T;H^{\gamma+1}(U))} \leq C_1 \left\| g_t \right\|_{L^2(0,t;H^{\gamma+1}(U))}.
\]

(3) Now, we shall prove that \( F \) is a compact linear operator.

Let \( g_t \) be a bounded sequence in \( W^{1,2}(0,T;H^{-1+1}(U)) \cap L^2(0,T;H^{1+1}(U)) \). Let \( 0 < \beta \leq \gamma - 1 \) when \( \alpha = 1 \) and \( 0 < \beta \leq \gamma + 1 \) when \( \alpha = 0 \). By the Banach-Alaoglu theorem and Canton diagonalization argument, \( g_t \) have a weakly convergence subsequence \( g_t \) such that (a) \( D^{\beta-1} \frac{\partial^\alpha}{\partial t^\alpha} g_t(t, x) \) is a weakly convergence subsequence in \( L^2(0,t;L^2(U)) \) and (b) \( D^\beta g_t(0;x) \) weakly converge in \( L^2(U) \). i.e. for any \( \varepsilon > 0 \), there exists a \( l_\varepsilon \), when \( l_1, l_2 > l_\varepsilon \).

\[
\left\| D^{\beta-1} \frac{\partial^\alpha}{\partial t^\alpha} (g_{l_1} - g_{l_2}) \right\|_{L^2(0,t;L^2(U))} < c_4 \varepsilon \quad (3.16)
\]

and

\[
\left\| D^\beta (g_{l_1} - g_{l_2})(0;x) \right\|_{L^2(U)} < c_4 \varepsilon \quad (3.17)
\]
Here, $c_4$ is a constant. Now, denote $\Delta g_t$ by $g_{t_1} - g_{t_2}$. The inequality (A.33) follows

$$\|F(\Delta g)\|_{L^3((0,T;H^r(U))} \leq C_3 \|\Delta g\|_{L^2((0,T;H^{r+1}(U))}$$

The inequality (A.32) implies that

$$\|F(\Delta g_t)\|_{L^3((0,T;H^r(U))} \leq C_1 \|\Delta g_t\|_{W^{1,2}((0,T;H^{r-1}(U))} + C_2 \|\Delta g_t(0; x)\|_{H^r(U)}.$$

Above inequalities together with (A.37) and (A.38) indicate that $F$ is a compact linear operator.
CHAPTER 4. SUMMARY AND CONCLUSION

We prove that these space are Banack space. Although $C^s([0,T])$ and $W^{s,\infty}([0,T])$ are not Banach space, this thesis proves that the combination of $C^s([0,T])$ and $H^m(U)$, $C^s([0,T]) \times H^m(U)$, is a Banach space if the boundary of $U$ satisfies the cone condition and $m > N/2$. Two functional spaces, $\mathcal{N}^{s+1,2}_{m+1}(0,T;U)$ and $\mathcal{Z}^s(0,T;H^m(U))$ are created by this thesis. In these space, the norm of the multiplication of two functions are bounded by the multiplication of the norm of these functions. This property enables us to bound the polynomial of linear operator. Thus, the nonlinear operator can be approached by a series of linear operator. We construct a Cauchy sequence which converge to the solutions.

In the append, we prove that this paper proposes a monotonic method to approach the strong solutions of nonlinear parabolic system (A.1)-(A.2). By this method, we prove that if the initial data satisfies inequality (A.67), then the global solution of system (A.1)-(A.2) exists in $\mathcal{N}^{s+1,2}_{m+1}(0,T;U)$. Otherwise, there exists a $T^*$ such that the solution exists in $\mathcal{N}^{s+1,2}_{m+2,2}(0,T^*;U)$. And theorem 19 gives a time interval that the solution will not blow up.
APPENDIX A. THE EXISTENCE OF THE GLOBAL AND LOCAL SOLUTION IN $\mathcal{K}_{m+1}^{s+1,2}(0, T; U)$

We study the Cauchy-Dirichlet problem of strong solution for a class of nonlinear parabolic partial difference equations

\[ \frac{\partial u}{\partial t} + f(u, x, Du, D^2U) = f_0 \]  \hspace{1cm} (A.1)
\[ u(0, U) = g(U), \quad u(t, \partial U) = h(t, \partial U), \]  \hspace{1cm} (A.2)

in the domain $[0, T] \times U$, where the nonlinear function $f$ is a polynomial of linear operators and can be represented by

\[ f(u, x, Du, D^2U) = Lu - L_1 u. \]  \hspace{1cm} (A.3)

Here, $Lu$ is a linear part of $f$ obeying

\[ Lu := -D_i(a^{ij}(x)D_j u) + b(x)u, \]  \hspace{1cm} (A.4)

and $L_1 u$ is the high order part of $f$ given by

\[ L_1 u = \sum_{i=2}^{M} \sum_{n=1}^{N} \prod_{j=1}^{i} F_{i,j,n}(u). \]  \hspace{1cm} (A.5)

We assume that the Dirichlet problem of the system (A.1)-(A.2) hold following conditions:

(A1), $U$ is a bounded connected open set in $\mathbb{R}^N$ satisfying $C^{m+2s+2,1}$-regularity property, and there exists a $m + 2s + 2$-smooth one to one transform of $U$ onto disk$(0, 1)$, where $\infty > m > N/2$, $\infty > s > 0$, and $N$ is a positive integer.

(A2), $a^{ij}(x) \in C^{m+2s+1,1}(\bar{U})$ and $b(x)$ are non-negative. L is self-adjoint, bounded and strong elliptic, i.e. there exist positive number $\gamma_1, \gamma_2$, and a non-negative number $\gamma_3$ such
that \( a^{ij} = a^{ji} \).

\[
|a^{ij}(x)\xi_i\xi_j| \geq \gamma_1 |\xi|^2, \quad \forall x \in U, \xi \in \mathbb{R}^N, \tag{A.6}
\]

\[
\sum |a^{ij}(x)|^2 \leq \gamma_2, \quad \forall x \in U, \tag{A.7}
\]

and

\[
\infty > b(x) \geq \gamma_3 \geq 0, \quad \forall x \in U. \tag{A.8}
\]

Moreover, \( g \in H^{m+2s+1}(U) \), \( h \in \mathcal{F}^1(0,T;H^{m+2s+1}(\partial U)) \cap \mathcal{F}^0(0,T;H^{m+2s+3}(\partial U)) \), and \( f_0 \in H^{m+2s}(U) \) is a force independent of \( u \).

(A3), each \( F_{i,j,n} \) is a linear operator from \( \mathcal{F}^j(0,T;H^{a+p_{i,j,n}(U)}) \) into \( \mathcal{F}^j(0,T;H^a(U)) \) so that

\[
\|F_{i,j,n}(u)\|_{\mathcal{F}^j(0,T;H^a(U))} \leq a_{i,j,n} \|u\|_{\mathcal{F}^j(0,T;H^{a+p_{i,j,n}(U)})} \tag{A.9}
\]

where \( m + 2s + 1 \geq \alpha \geq 0 \), \( s \geq j \geq 0 \), \( 1 \geq p_{i,j,n} > -\infty \), and the space \( \mathcal{F}^j(0,T;H^a(U)) \) is defined by definition 11.

\( f \) is uniformly elliptic, but may not be convex or proper because of the condition (A2) and (A3). When the condition (A1) and (A2) are held, the eigenvalues of operator \( Lu \) are positive and the eigenfunctions \( \varphi_{\lambda_i} \in H^1_0(U) \cap H^{m+2s+2}(U) \) span a Hilbert space \( H(U) \) (theorem 8.37, Gilbarg and Trudinger (2001)). Moreover, \( F_{i,j,n} \) can be convolution, scaler multiplication, differential or integral operator etc. Nonlinear parabolic equation (A.1) is a general representation of reaction-diffusion equation and Burger’s equation etc. Here, the \( i \) is the order of the nonlinear operator \( \prod_{j=1}^i F_{i,j,n} \) and the \( N_i \) is the maximum number of the nonlinear operator \( \prod_{j=1}^i F_{i,j,n} \) with the order \( i \).

There are some dramatic efforts carried out in the study of the fully nonlinear partial difference equations including the elliptic and parabolic types during the last three decades. Two main methods have been developed for solving fully nonlinear elliptic equations. One approach is to prove the existence of classical solutions of the Dirichlet problem in a smooth bounded domain \( U \in \mathbb{R}^N \) directly using the continuity method (Evans (1982), Krylov (1983), and Caffarelli and Huang (2003)). For this, one needs to prove a priori estimates for solutions
in the space $C^{2,\alpha}(\bar{U})$ for some $0 < \alpha < 1$. The second approach called viscosity solution method is to prove the existence of some sort of generalized solutions and then to achieve their uniqueness and regularity (Evans (1978), Crandall and Lions (1983), and Crandall et al. (1992)). A major breakthrough in the theory of viscosity solutions was made by Jensen Jensen (1988), who proved a comparison principle which turned out the uniqueness of viscosity solutions of the Dirichlet problem for

$$f(D^2 u, x) = 0$$

at least for $f$ independent of $x$, but the existence of solutions of the Dirichlet problem is not proved. Indeed, our knowledge about the existence and high order regularity of fully nonlinear PDEs remains elusive, particularly for the nonlinear PDEs which are not convex or proper. A typical case is the Navier-Stokes equation. Although the existence of the weak solution (Leray (1934), Caffarelli et al. (1982)) and mild solution Kato (1984) of Navier-Stokes equation are known, the global existence of the strong solution still remains open.

The objective of this paper is to prove the Dirichlet problem of the strong global solution in the space $[0, T] \times U$ for a class of nonlinear parabolic systems (A.1)-(A.2) under the condition (A1), (A2), and (A3), where the system may not be convex.

### A.1 Notation and the main ideas

#### A.1.1 Notation

**Definition 11** Denote $\mathcal{S}^s(0, T; H^m(U))$ by a vector space consists of all functions $u$ such that

$$|u|_{\mathcal{S}^s(0, T; H^m(U))} := \sum_{0 \leq j \leq s} \max_{0 \leq t \leq T} \|\frac{\partial^j u}{\partial t^j}\|_{H^m(U)} < \infty$$

where $U \in \mathbb{R}^N$.

**Definition 12** Denote $\mathcal{K}^s_m(0, T; U)$ by a vector space consists of all functions $u$,

$$u \in \cap_{j=0}^s \mathcal{S}^{s-j}(0, T; H^{m+q}(U))$$

(A.10)
where \( U \in \mathbb{R}^N \) and \( q \geq 2 \). \( \| u \|_{\mathcal{H}^m(0, T; H^m(U))} \) is defined by
\[
\| u \|_{\mathcal{H}^m(0, T; H^m(U))} = \sum_{0 < j < s} \| u \|_{\mathcal{H}^{s-j}(0, T; H^{m+qj}(U))}
\]

**Definition 13** Let the constant \( \varepsilon^* \) be arbitrary in the domain \([\frac{1}{4}, \frac{1}{2}]\), \( \theta = \frac{4C_{\gamma}}{\varepsilon^*} \),
\[
C_{\gamma} = 2N^{m+2(s+1)} \max\{1, \frac{\sqrt{T}}{\sqrt{\gamma_1}}, \min\{T, \frac{\sqrt{T}}{\sqrt{\gamma_3}}\}\},
\]
\[
C_{\theta} = M^2 C_{\theta}^M \sum_{i=2}^{M} \sum_{n=1}^{N_i} [(i - 1) \prod_{l=1}^{1} (a_{i,l,n})],
\]
\[
\lambda = \min\{1, \frac{\varepsilon^*}{4MC_{\gamma}\max\{1, C_{\theta}^M\} \sum_{i=2}^{M} \sum_{n=1}^{N_i} (\prod_{l=1}^{1} a_{i,l,n})}\}, \tag{A.11}
\]
and
\[
\beta = \min\{1, \frac{(1 - \varepsilon^*)}{2\theta \| r_0 \|_{\mathcal{H}^{s+1,2}_m(0, T; U)}}, \frac{(1 - \varepsilon^*)^2}{4C_{\theta}^2 \theta \| r_0 \|_{\mathcal{H}^{s+1,2}_m(0, T; U)}}\} \tag{A.12}
\]
respectively. Here, \( C_* \) is a constant given by lemma 13, \( r_0 \) is given by A.13.

### A.1.2 The idea of construction

The main idea of this paper is to construct a Cauchy sequence \( \{u_k\} \) to approach the global solution of nonlinear parabolic equation (A.1) in the space \( \mathcal{H}^{s+1,2}_m(0, T; U) \). The proposition 11 and 12 will show that \( \mathcal{H}^{s+1,2}_m(0, T; U) \) is complete.

Assume that \( u_k \) is not the solution. It must have a nontrivial residue \( r_k \), where \( r_k \) is defined by
\[
r_k := (\frac{\partial}{\partial t} + L)u_k - f_0 - L_1 u_k. \tag{A.13}
\]

Let \( \rho_k s_k := u_{k+1} - u_k \), where \( \rho_k \) is a constant less than 1. Since \( f(u_{k+1}) \) is a polynomial of linear functional on \( u_{k+1} \), identity (A.1) can be decomposed into three parts;
\[
r_{k+1} = r_k + F_1(\rho_k s_k) + F_2(\rho_k s_k) \tag{A.14}
\]
where \( F_1(\rho_k s_k) \) is given by
\[
F_1(\rho s_k) := (\frac{\partial}{\partial t} + L)(\rho_k s_k) - \sum_{i=2}^{M} \sum_{n=1}^{N_i} \sum_{j=1}^{i} F_{i,j,n}(\rho_k s_k) (\prod_{l=1,l \neq j}^{i-1} F_{i,l,n}(u_k)) \tag{A.15}
\]
and $F_2(\rho_k s_k)$ is denoted by

$$F_2(\rho_k s_k) = r_{k+1} - r_k - F_1(\rho_k s_k)$$

$$= - \sum_{i=2}^{M} \sum_{j=2}^{i} \left( \frac{\alpha}{j} \right) \left( \prod_{l=1}^{j} F_{i,j_l,n}(\rho_k s_k) \right) \left( \prod_{l=1}^{i-j} F_{i,j_l,n}(u_k) \right)$$

where $j_l$ and $j_l$ form a partition of set $\{1, 2, ..., i\}$. Clearly, $F_1(\rho_k s_k)$ is a linear operator of $\rho s_k$, more precisely,

$$F_1(\rho s_k) = \rho_k F_1(s_k)$$

And $F_2(\rho s_k)$ is a nonlinear operator of $\rho s_k$ but is homogenous with order larger than 1.

In order to achieve the monotonic convergence of the residue $r_k$, we want

$$|r_{k+1}|_{W_{m+2s}^{1,2}(0,T;U)} \leq (1 - \beta) \|r_k\|_{W_{m+2s}^{1,2}(0,T;U)}$$

(A.17)

for some fixed constant $\beta \in (0, 1)$. Since we want $\{u_k\}$ to be Cauchy, we attempt to control the step length, $\rho_k s_k$, by

$$\|\rho_k s_k\|_{W_{m+2s+1}^{1,2}(0,T;U)} \leq \rho_k \theta \|r_k\|_{W_{m+2s}^{1,2}(0,T;U)}$$

(A.18)

where $\theta$ is independent of $k$ and $0 < \rho_k \leq 1$.

However, there may not be an $s_k$ such that

$$F_1(s_k) = -r_k$$

(A.19)

The idea here is to seek a perturbation $R_k$ of $r_k$ such that

$$\|R_k - r_k\|_{W_{m+2s}^{1,2}(0,T;U)} \leq \varepsilon^* \|r_k\|_{W_{m+2s}^{1,2}(0,T;U)}$$

(A.20)

for some small $\varepsilon^*$, and there exists an $s_k$ such that

$$F_1(s_k) = -R_k.$$ 

(A.21)

The method to construct such a $R_k$ is proposed by theorem 16. This theorem proves the existence of $R_k$ and $s_k$ and the regularity of the solution $\{s_k\}$.

In conclusion, we perturb $r_k$ such that equation (A.21) has a solution satisfying condition (A.17) and (A.18). The inequality (A.17) and (A.18) will guarantee the convergence of the Cauchy sequence $\{u_k\}$ in $W_{m+2s+1}^{1,2}(0,T;U)$ for any $T$. 
A.2 Main Theorems

A.2.1 Preliminaries

We will show that the space $\mathcal{S}^s(0, T; H^m(U))$ and $\mathcal{N}_m^{a,q}(0, T; U)$ are the Banach space for any finite $s, m \geq 0$ and $q \geq 2$.

**Theorem 11** The space $\mathcal{S}^s(0, T; H^m(U))$ is a Banach space if $U \subset \mathbb{R}^N$ satisfies the cone condition and $m > N/2$.

**Proof.** (1) Clearly, $\mathcal{S}^s(0, T; H^m(U))$ is a norm space. We will show next that it is complete. Suppose that $\{u_n\}$ is Cauchy in $\mathcal{S}^s(0, T; H^m(U))$. For every given $t$, $\{\frac{\partial^j u_n}{\partial^j t^j}\}$ is Cauchy in $H^m(U)$, where $0 \leq j \leq s$, because for any pair of $u_n$, and $u_{n'}$, we have

$$
\left\| \frac{\partial^j u_{n_1}}{\partial^j t^j} - \frac{\partial^j u_{n_2}}{\partial^j t^j} \right\|_{H^m(U)}(t) \leq \max_{t \in [0, T]} \left\| \frac{\partial^j u_{n_1}}{\partial^j t^j} - \frac{\partial^j u_{n_2}}{\partial^j t^j} \right\|_{H^m(U)} \\
\leq \|u_{n_1} - u_{n_2}\|_{\mathcal{S}^s(0, T; H^m(U))},
$$

Considering that $H^m(U)$ is closed, for any given $t$, $\frac{\partial^j u_n}{\partial^j t^j}$ converges to function $v_j(t, x)$ in $\mathcal{S}^0(0, T; H^m(U))$. It is easy to show that $v_j(t, x)$ is well defined and $v_j \in \mathcal{S}^0(0, T; H^m(U))$. By the imbedding theorem, $H^m(U) \rightarrow C^r(U)$ where $r$ is the largest integer less than $m - N/2$. i.e., there exist a constant $k$ such that

$$
\left\| \frac{\partial^j u_{n_1}}{\partial^j t^j} - \frac{\partial^j u_{n_2}}{\partial^j t^j} \right\|_{C^r(U)}(t) \leq k \left\| \frac{\partial^j u_{n_1}}{\partial^j t^j} - \frac{\partial^j u_{n_2}}{\partial^j t^j} \right\|_{H^m(U)}(t) \quad (A.22)
$$

where $k$ is independent of $j$, $n$, and $t$. Thus, $\{u_n\}$ uniformly converge in $\mathcal{S}^0(0, T; C^r_B(U))$ and $u_n, v_j \in \mathcal{S}^0(0, T; C_{B}^r(U))$ for any $0 \leq j \leq s$.

(2) In this step, we will show that $v_j(t, x) = \frac{\partial^{j-1} u_n}{\partial t^{j-1}}$ in the space $\mathcal{S}^0(0, T; C^r_B(U))$ for any $0 < j \leq s$. At first, we will prove that $\frac{\partial^{j-1} u_n}{\partial t^{j-1}}$ and $\frac{\partial^{j-1} u_{n'}}{\partial t^{j-1}}$ are uniformly converge to $v_{j-1}(t, x)$ and $v_j(t, x)$ in $[0, T] \times U$. By the definition of norm,

$$
\max_{|\alpha| \leq j} \sup_{x \in U} \left\| \frac{\partial^{|\alpha|} u_{n_1}}{\partial t^{|\alpha|}}(t, x) - \frac{\partial^{|\alpha|} u_{n_2}}{\partial t^{|\alpha|}}(t, x) \right\| = \left\| \frac{\partial^j u_{n_1}}{\partial^j t^j} - \frac{\partial^j u_{n_2}}{\partial^j t^j} \right\|_{C^r(U)}(t) \leq k \|u_{n_1} - u_{n_2}\|_{\mathcal{S}^s(0, T; H^m(U))}
$$
Considering that sequence \( \{ u_n \} \) is Cauchy, we have
\[
\lim_{n \to \infty} \left| \frac{\partial^n u_n}{\partial t^n} - v_j \right| (t, x) \leq k \| u_{n_1} - v_j \| _{\mathfrak{A}^s(0,T;H^m(U))} \tag{A.23}
\]
for every \( x \) and \( t \). This proves that the sequence \( \{ \frac{\partial^n u_n}{\partial t^n} \} \) and \( \{ \frac{\partial^n u_n}{\partial t^n} \} \) converge to \( v_{j-1}(t, x) \) and \( v_j(t, x) \) uniformly in \([0,T] \times U\).

For every given \( x \) in \( U \), by the uniform convergence and differentiation theorem (theorem 7.17, Rudin (1976)), we have \( \frac{\partial}{\partial t} v_{j-1}(t, x) = v_j(t, x) \) for this given \( x \). Since \( x \) is arbitrary, we conclude that \( \frac{\partial}{\partial t} v_{j-1}(t, x) = v_j(t, x) \) in \([0,T] \times U\). Moreover, we have \( \frac{\partial}{\partial t} v_{j-1}(t, x) = v_j(t, x) \) in \( \mathfrak{A}^0(0,T;H^m(U)) \).

(3) Now, we are going to show that \( v_j(t, x) = \frac{\partial v_{j-1}}{\partial t} \) in the space \( \mathfrak{A}^0(0,T;H^m(U)) \) for any \( 0 < j \leq s \). By the idea of induction, we assume \( v_j(t, x) = \frac{\partial v_{j-1}}{\partial t} \) in the space \( \mathfrak{A}^0(0,T;H^r(U)) \) where \( 0 < j \leq s \) and \( r < m \). Let \( \Phi \) be a test function, thus
\[
\left( \frac{\partial v_{j-1}}{\partial t} - v_j(t, x), D^{r+1}\Phi \right) = \lim_{n \to \infty} \left( \frac{\partial}{\partial t} \frac{\partial^{j-1} u_n}{\partial t^{j-1}} - \frac{\partial^{j} u_n}{\partial t^{j}}, D^{r+1}\Phi \right)
= \lim_{n \to \infty} (-1)^{r+1} \left( D^{r+1} \frac{\partial}{\partial t} \frac{\partial^{j-1} u_n}{\partial t^{j-1}} - D^{r+1} \frac{\partial^{j} u_n}{\partial t^{j}} \Phi \right)
= 0.
\]
Considering that \( j \) and \( r \) are arbitrary, we complete the proof. \( \blacksquare \)

We know that \( C^s([0,T]) \) is not Banach space, so does \( W^{s,\infty}([0,T]) \). However, the combination of \( C^s([0,T]) \) and \( H^m(U) \), \( C^s([0,T]) \times H^m(U) \), is a Banach space if the boundary of \( U \) satisfies the cone condition and \( m > N/2 \). This is not clear whether the theorem 11 can be satisfied for more general domain \( U \) or not.

**Theorem 12** The space \( \mathfrak{A}^q_{m,0}(0,T;U) \) is a Banach space if \( U \subset \mathbb{R}^N \) satisfies the cone condition and \( m > N/2 \).

**Proof.** Suppose sequence \( \{ u_n \} \) is Cauchy in \( \mathfrak{A}^q_{m,0}(0,T;U) \). By theorem 11, sequence \( \{ u_n \} \) converges to \( v_{s-j} \) in \( \mathfrak{A}^{s-j}(0,T;H^{m+qj}(U)) \) for each \( 0 \leq j \leq s \). To prove the proposition, it suffices to show that \( v_{s-j} \in \mathfrak{A}^{s-j-1}(0,T;H^{m+qj+q}(U)) \) for each \( 0 \leq j \leq s \). Since \( \mathfrak{A}^{s-j}(0,T;H^{m+qj}(U)) \subset \mathfrak{A}^{s-j-1}(0,T;H^{m+qj}(U)) \), we obtain
\[
v_{s-j} \in \mathfrak{A}^{s-j-1}(0,T;H^{m+qj}(U))
\]
Thus, for any \( t \) and \( k \), where \( 0 \leq k \leq s - j - 1 \), \( \frac{\partial^k u_n}{\partial t^k} \) converges to \( \frac{\partial^k u_{s-j}}{\partial t^k} \) in \( H^{m+qj}(U) \) uniformly. Since \( \{u_n\} \) is Cauchy in \( \mathbb{R}^{s,q}(0,T;U) \) and \( q \geq 1 \), sequence \( \{\frac{\partial}{\partial x} \frac{\partial^k u_n}{\partial t^k}\} \) converge in \( H^{m+qj}(U) \) uniformly. For any \( 0 \leq k \leq s - j - 1 \) and \( 0 \leq r \leq m + qj + q \),

\[
\left( \frac{\partial^k u_{s-j}}{\partial t^k}, D^r \Phi \right) = \lim_n \left( \frac{\partial^k u_n}{\partial t^k}, D^r \Phi \right) = \lim_n (-1)^r \left( D^r \frac{\partial^k u_n}{\partial t^k}, \Phi \right).
\]

We know that \( \lim_n D^r \frac{\partial^k u_n}{\partial t^k} \) exists in \( H^{m+qj+q} \) for each \( t \) and \( k \). Thus,

\[ v_{s-j} \in \mathbb{R}^{s-j-1}(0,T;H^{m+qj+q}(U)). \]

This completes the proof. ■

As a prelude to existence considerations we derive now some lemmas.

**Lemma 13** Let \( u_1, u_2 \in \mathbb{R}^{s}(0,T;W^{m,p}(U)) \) and \( u_3, u_4 \in \mathbb{R}^{s,q}(0,T;U) \). Let \( U \) be a set in \( \mathbb{R}^N \) satisfying a cone property. For each \( \infty > pm > N \), \( q > 1 \), and \( \infty > s \geq 0 \), there exists a constant \( C_* = C_*(N,U,m,p,s) \) such that

\[
\|u_1 u_2\|_{\mathbb{R}^s(0,T;W^{m,p}(U))} \leq C_* \|u_1\|_{\mathbb{R}^s(0,T;W^{m,p}(U))} \|u_2\|_{\mathbb{R}^s(0,T;W^{m,p}(U))}; \quad (A.24)
\]

\[
\|u_1 u_2\|_{W^{s,\infty}(0,T;W^{m,p}(U))} \leq C_* \|u_1\|_{W^{s,\infty}(0,T;W^{m,p}(U))} \|u_2\|_{W^{s,\infty}(0,T;W^{m,p}(U))}; \quad (A.25)
\]

and

\[
\|u_3 u_4\|_{\mathbb{R}^{s,q}(0,T;U)} \leq C_* \|u_3\|_{\mathbb{R}^{s,q}(0,T;U)} \|u_4\|_{\mathbb{R}^{s,q}(0,T;U)}; \quad (A.26)
\]

**Proof.** Suppose \( u_1, u_2 \in \mathbb{R}^{s}(0,T;W^{m,p}(U)) \). Let \( \gamma \) be such that \( s \geq \gamma \geq 0 \), then \( \frac{\partial^\gamma u_1}{\partial t^\gamma}, \frac{\partial^{s-\gamma} u_2}{\partial t^{s-\gamma}} \in W^{m,p}(U) \). Note that \( U \) holds a cone property and \( \infty > pm > N \). Due to the theorem 4.39 of Adams (2003), there exists a constant \( c_1 \) such that for any \( pm > N \),

\[
\left\| \frac{\partial^\gamma u_1}{\partial t^\gamma} \frac{\partial^{s-\gamma} u_2}{\partial t^{s-\gamma}} \right\|_{W^{m,p}(U)} \leq c_1(U,N,m,p) \left\| \frac{\partial^\gamma u_1}{\partial t^\gamma} \right\|_{W^{m,p}(U)} \left\| \frac{\partial^{s-\gamma} u_2}{\partial t^{s-\gamma}} \right\|_{W^{m,p}(U)}
\]

where \( c_1 \) only depends \( U, N, m \) and \( p \).
The boundedness of \( u_1 u_2 \) in \( \mathfrak{H}^s(0,T;W^{m,p}(U)) \) follows:

\[
\|u_1 u_2\|_{\mathfrak{H}^s(0,T;W^{m,p}(U))} = \max_{0 \leq t \leq T} \left\| \frac{\partial^s}{\partial t^s} u_1 u_2 \right\|_{W^{m,p}(U)} \\
\leq \max_{0 \leq t \leq T} \sum_{\gamma=0}^{s} \binom{s}{\gamma} c_1 \left\| \frac{\partial^\gamma u_1}{\partial t^\gamma} \right\|_{W^{m,p}(U)} \left\| \frac{\partial^{s-\gamma} u_2}{\partial t^{s-\gamma}} \right\|_{W^{m,p}(U)}.
\]

Note that for each given \( t \in [0,T] \), \( \left\| \frac{\partial^\gamma u_1}{\partial t^\gamma} \right\|_{W^{m,p}(U)} \) and \( \left\| \frac{\partial^{s-\gamma} u_2}{\partial t^{s-\gamma}} \right\|_{W^{m,p}(U)} \) are in \( L^\infty([0,T]) \).

Therefore, \( \|u_1 u_2\|_{\mathfrak{H}^s(0,T;W^{m,p}(U))} \) is bounded. Furthermore, there exists a constant \( C'_s \), which is independent of \( T \), such that

\[
\|u_1 u_2\|_{\mathfrak{H}^s(0,T;W^{m,p}(U))} \leq C'_s \|u_1\|_{\mathfrak{H}^s(0,T;W^{m,p}(U))} \left\|u_2\right\|_{\mathfrak{H}^s(0,T;W^{m,p}(U))}.
\]

By repeating above deduction, we conclude that for each \( 0 \leq j \leq s \), there exists a constant \( c_j \) such that

\[
\|u_3 u_4\|_{\mathfrak{H}^{s-j}(0,T;W^{m+q,j,p}(U))} \leq c_j \left\|u_3\right\|_{\mathfrak{H}^{s-j}(0,T;W^{m+q,j,p}(U))} \left\|u_4\right\|_{\mathfrak{H}^{s-j}(0,T;W^{m+q,j,p}(U))}.
\]

Let \( C_s = \max(c_j, c, C'_s) \), we obtain:

\[
\|u_3 u_4\|_{\mathfrak{K}^{s,q}_{m,n}(0,T;U)} \leq C_s \left\|u_3\right\|_{\mathfrak{K}^{s,q}_{m,n}(0,T;U)} \left\|u_4\right\|_{\mathfrak{K}^{s,q}_{m,n}(0,T;U)}
\]

and

\[
\|u_1 u_2\|_{\mathfrak{H}^s(0,T;W^{m,p}(U))} \leq C_s \|u_1\|_{\mathfrak{H}^s(0,T;W^{m,p}(U))} \left\|u_2\right\|_{\mathfrak{H}^s(0,T;W^{m,p}(U))}.
\]

In the same way, we can prove the inequality (A.25). \( \blacksquare \)

Preceding lemma enables us to bound the norm of a polynomial by the corresponding polynomial of norm. With these inequalities, we can easily prove that the nonlinear operator \( F_2 \) is bounded by \( r_k \).

Remark 2 Suppose \( \varphi_i \) to be the normalized eigenfunctions of the Dirichlet problem for the operator \( Lu \) denoted by (A.4). The solution of generated Dirichlet problem of the heat equation

\[
\begin{align*}
&u_t + Lu = f, \\
&u(0,U) = g(U), \quad u(t,\partial U) = h(t; \partial U),
\end{align*}
\]
is given by

$$u = \sum_{i=1}^{\infty} e^{-\lambda_i t} \varphi_i \left( \int_U g_i \varphi_i + \int_0^t e^{\lambda_i \tau} \int_U \varphi_i(\xi) f(\tau; \xi) d\xi d\tau - \int_0^t \int_{\partial U} h \frac{\partial \varphi_i(\xi)}{\partial \xi_n} d\xi d\tau \right).$$

Denote

$$\psi(t; x|\tau; \xi) = \sum_{i=1}^{\binom{N}{2}} e^{-\lambda_i (t - \tau)} \varphi_i(x) \varphi_i(\xi),$$

(A.27)

It yields

$$u = \int_0^t \int_U \psi(t; x|\tau; \xi) f(\tau; \xi) d\xi d\tau + \int_U \psi(t; x|0; \xi) g(\xi) d\xi d\tau - \int_0^t \int_{\partial U} h \frac{\partial \psi}{\partial \xi_n}.$$

Here, $\psi(t; x|\tau; \xi)$ is the Dirichlet heat kernel. The regularity of $\psi(t; x|\tau; \xi)$ depends on the smoothness of the domain $U$ and the coefficients of operator $L$.

**Lemma 14** Under condition (A1) and (A2), the operator $Lu$ has a set of eigenfunctions $\{\varphi_{\lambda_i}\}$ obeying (1) $\{\varphi_{\lambda_i}\}$ form an orthonormal basis of a Hilbert space $H(U)$, (2) $\varphi_{\lambda_i} \in H_0^\beta(U)$ and the eigenvalues $\lambda_i$ hold

$$0 < \lambda_1 < ... < \lambda_n < ..., \quad (A.28)$$

and (3) $0 \leq \beta \leq m + 2(1 + s)$ and $\varphi_i, \varphi_j \in \{\varphi_{\lambda_i}\}$,

$$\left\langle D^\beta \varphi_i, D^\beta \varphi_j \right\rangle = 0, \quad \text{if } i \neq j \quad (A.29)$$

$$\left\langle D^\beta \varphi_i, D^\beta \varphi_i \right\rangle \leq \frac{\lambda_i - \gamma_3}{\gamma_1} \left\langle D^{\beta - 1} \varphi_i, D^{\beta - 1} \varphi_i \right\rangle \quad (A.30)$$

where $\gamma_1$ is given by condition (A2).

**Proof.** It is known that the eigenfunctions $\varphi_{\lambda_i}^1 \in H^{m+2(s+1)}(U) \cap H_0^1(U)$ of operator $Lu$ span a Hilbert space $H(U)$, and the corresponding eigenvalues hold (A.28) (Theorem 8.37 and 9.19, Gilbarg and Trudinger (2001)). Denote $\varphi_{\lambda_i} = \lim_{\varepsilon \to 0} \eta_{\varepsilon} \otimes \varphi_{\lambda_i}^2$, where $\eta_{\varepsilon}$ is a standard mollifier and

$$\varphi_{\lambda_i}^2 = \varphi_{\lambda_i}, \quad \text{when } x \in U,$$

$$\varphi_{\lambda_i}^2 = 0, \quad \text{when } x \in \mathbb{R}^N \setminus U.$$

Since $U$ is a bounded open set in $\mathbb{R}^N$, $\varphi_{\lambda_i} \in H^{m+2(s+1)}(U)$ and $\varphi_{\lambda_i} = \varphi_{\lambda_i}^1$ in $H^{m+2(s+1)}(U) \cap H_0^1(U)$. Then, $\{\varphi_{\lambda_i}\}$ span $H(U)$. Meanwhile, eigenvalues of $\varphi_{\lambda_i}$ is equivalent to that of $\varphi_{\lambda_i}^1$. 

Thus, (A.28) is satisfied. The orthogonality of \( \{ \varphi_{\lambda_i} \} \) in \( H(U) \) immediately follows the orthogonality of \( \{ \varphi_{\lambda_i} \} \) in \( H(U) \). The reminded problem is to show the orthogonality of \( \{ D^\beta \varphi_{\lambda_i} \} \) for any \( 0 < \beta \leq m + 2(s + 1) \). Suppose that \( \varphi_i, \varphi_j \) are arbitrary in \( \{ \varphi_{\lambda_i} \} \) and \( i \neq j \). The Green’s first identity Gilbarg and Trudinger (2001) and the facts \( \varphi_i, \varphi_j \in H_{0}^{m+2(s+1)}(\overline{U}) \) immediately follow

\[
\lambda_j \left< D^\beta \varphi_i, D^\beta \varphi_j \right> = \left< D^\beta \varphi_i, LD^\beta \varphi_j \right> \\
= \left< \alpha ij D^\beta \varphi_i, D^\beta \varphi_j \right> + \left< b D^\beta \varphi_i, D^\beta \varphi_j \right>
\]

where operator \( L \) is defined by the condition (A.2). Together with the strong elliptic condition (A.6)-(A.8), we obtain:

\[
\lambda_j \left< D^\beta \varphi_i, D^\beta \varphi_j \right> \geq \gamma_1 \left< D^\beta \varphi_i, D^\beta \varphi_j \right> + \gamma_3 \left< D^\beta \varphi_i, D^\beta \varphi_j \right>
\]

It is clear that \( \lambda_j > \gamma_3 \) because

\[
\lambda_j \geq \min_{||u||=1, u \in H_0^1(U)} \langle u, Lu \rangle \geq \min_{||u||=1, u \in H_0^1(U)} \gamma_1 \||D u|| + \gamma_3.
\]

Thus, \( \langle D^\beta \varphi_i, D^\beta \varphi_j \rangle \leq \frac{\lambda_j - \gamma_3}{\gamma_1} \langle D^\beta \varphi_i, D^\beta \varphi_j \rangle = \left( \frac{\lambda_j - \gamma_3}{\gamma_1} \right)^{\beta} \langle \varphi_i, \varphi_j \rangle \). This proves the inequality (A.30).

The following theorem deals with the upper bound of the heat kernel.

**Theorem 15** Denote \( F(g) := \int_0^T \int_U \psi(0; t; x) g(t - \tau; y) (t - \tau; y) d\tau d\xi \) and let \( X_1 \) to be the space \( W^{1,2}(0, T; \mathcal{H}_r^{-1}(U)) \cap L^2(0, T; \mathcal{H}_r^{s+1}(U)) \cap L^\infty(0, H^r(U)) \). Under condition (A1) and (A2), for any \( \gamma \geq 1 \), if \( g \in X_1 \), then

1. there exist follow inequalities:

\[
\| F(g) \|_{\mathcal{H}_r(0, T; \mathcal{H}_r(U))} \leq C_1 \| g \|_{W^{1,2}(0, t; \mathcal{H}_r^{-1}(U))} + C_2 \| g(0; t) \|_{H^\gamma(U)}
\]

and

\[
\| F(g) \|_{\mathcal{H}_r(0, T; \mathcal{H}_r(U))} \leq C_1 \| g \|_{L^2(0, t; \mathcal{H}_r^{s+1}(U))}
\]

where \( C_1 = N^{\gamma+1} \max\left( \frac{1}{\sqrt{2\gamma_1}}, \min(\sqrt{T}, \frac{1}{\sqrt{2\gamma_3}}) \right) \) and \( C_2 = N^{\gamma+1} \).
(2) $F(g)$ is a compact linear operator from $X_1$ into $\mathcal{K}_{\gamma,2}(0; T; U)$.

**Proof.** (1) Firstly, we estimate the upper bound of $\|F(g)\|_{3^1(0; T; L^2(U))}$. It suffices to show that, for any $0 \leq \beta \leq \gamma$, $\|D^\beta \frac{\partial}{\partial t} F(g(t))\|_{3^0(0; T; L^2(U))}$ is bounded by $\|g\|_{W^{1,2}(0; T; H^{\gamma-1}(U))}$ and $\|g(t)\|_{L^2(0; T; H^{\gamma+1}(U))}$. According to the Leibnitz integral rule, we compute

$$\left\| D^\beta \frac{\partial}{\partial t} F(g(t)) \right\|_{L^2(U)} = \left\{ \int_U dx \left[ \frac{\partial}{\partial t} \int_0^t d\tau \int_U D^\beta \psi(0; x|\tau; y) g(t - \tau; y) dy \right]^2 \right\}^{1/2}$$

$$\leq \left\{ \int_U dx \left[ \int_0^t d\tau \int_U D^\beta \psi(0; x|\tau; y) \frac{\partial g(t - \tau; y)}{\partial t} dy \right]^2 \right\}^{1/2} + \left\{ \int_U dx \left[ \int_U D^\beta \psi(0; x|t; y) g(0; y) dy \right]^2 \right\}^{1/2}.$$

It follows from the identity (A.27),

$$\left\| D^\beta \frac{\partial}{\partial t} F(g(t)) \right\|_{L^2(U)} \leq \left\{ \int_U dx \left[ \int_0^t d\tau \sum_{i=1}^\infty e^{-\lambda_i \tau} D^\beta \varphi_i(x) \left\langle \varphi_i(\xi), \frac{\partial g(t)}{\partial t} \right\rangle \right]^2 \right\}^{1/2}$$

$$+ \left\{ \int_0^\infty \left[ \sum_{i=1}^\infty D^\beta \varphi_i(x) e^{-\lambda_i t} \left\langle \varphi_i(\xi), g(t; 0; y) \right\rangle \right]^2 \right\}^{1/2}$$

$$= : I + II. \quad (A.34)$$

Without loss of generality, we discuss the case $\beta \geq 1$ firstly. Due to the lemma 14, $D^\beta \varphi_i(x)$ are orthogonal. Thus,

$$I = \left\{ \sum_{i=1}^\infty \left\| D^\beta \varphi_i(x) \right\|_{L^2(U)}^2 \left[ \int_0^t e^{-\lambda_i \tau} \left\langle \varphi_i(\xi), \frac{\partial g(t)}{\partial t} \right\rangle \right]^2 \right\}^{1/2}.$$

The inequality (A.30) follows that

$$I \leq \left\{ \sum_{i=1}^\infty \left( \frac{\lambda_i - \gamma_3}{2 \gamma_i \lambda_i} \right) \left\| D^{\beta-1} \varphi_i(x) \right\|_{L^2(U)}^2 \left[ \int_0^t \sqrt{2\lambda_i e^{-\lambda_i \tau}} \left\langle \varphi_i(\xi), \frac{\partial g(t)}{\partial t} \right\rangle \right]^2 \right\}^{1/2}.$$ 

Clearly, $\frac{\lambda_i - \gamma_3}{2 \gamma_i \lambda_i} \leq \frac{1}{2 \gamma_i}$. By the Hölder’s inequality,

$$I \leq \left[ \sum_{i=1}^\infty \left\| D^{\beta-1} \varphi_i(x) \right\|_{L^2(U)}^2 \left[ \int_0^t 2\lambda_i e^{-2\lambda_i \tau} \int_0^t \left\langle \varphi_i(\xi), \frac{\partial g(t)}{\partial t} \right\rangle \right]^2 \right]^{1/2}$$

$$\leq \frac{1}{\sqrt{2 \gamma_i}} \left[ \int_0^t \sum_{i=1}^\infty \left\| D^{\beta-1} \varphi_i(x) \right\|_{L^2(U)}^2 \left\langle \varphi_i(\xi), \frac{\partial g(t)}{\partial t} \right\rangle \right]^{1/2}$$

$$= \frac{1}{\sqrt{2 \gamma_i}} \left\| D^{\beta-1} \frac{\partial g(t)}{\partial t} \right\|_{L^2(0; T; L^2(U))}$$

because $D^{\beta-1} \varphi_i(x)$ are orthogonal.
In the same way for $I$, we have

$$II = \sum_{i=1}^{\infty} \left\| D^\beta \varphi_i(x) \right\|_{L^2(U)}^2 e^{-2\lambda_i t} \left\| \varphi_i(x), g_i(0; y) \right\|_{L^2(U)}^{1/2}$$

(A.35)

$$\leq \left\| D^\beta g_i(0; y) \right\|_{L^2(U)}.$$

because $0 < e^{-2\lambda_i t} \leq 1$ and $D^\beta \varphi_i(x)$ are orthogonal. Thus, for any $t \in [0, T]$,

$$\left\| \frac{\partial}{\partial t} \frac{\partial^\beta}{\partial x^\beta} F(g_i) \right\|_{L^2(U)} \leq \frac{1}{\sqrt{2\gamma_1}} \left\| D^{\beta-1} \frac{\partial g_i}{\partial t} \right\|_{L^2(0, t; L^2(U))} + \left\| D^\beta g_i(0; y) \right\|_{L^2(U)}$$

(A.36)

For the case $|\beta| = 0$, by the analogous method with above, we deduced

$$\left\| \frac{\partial}{\partial t} F(g_i) \right\|_{L^2(U)} \leq \sum_{i=1}^{\infty} \left\| \varphi_i \right\|_{L^2(U)} \left\| \int_0^t e^{-\lambda_i \tau} \left\langle \varphi_i(x), \frac{\partial g_i}{\partial t} \right\rangle \right\|_{L^2(U)}^{1/2} + \left\| g_i(0; y) \right\|_{L^2(U)}$$

$$\leq \sum_{i=1}^{\infty} \left\| \varphi_i \right\|_{L^2(U)} \left\| \int_0^t e^{-2\lambda_i \tau} \right\|_{L^2(U)}^{1/2} + \left\| g_i(0; y) \right\|_{L^2(U)}.$$

In the lemma 14, we prove that $\lambda_i > \gamma_3$. Hence, $\sqrt{\int_0^T e^{-2\lambda_i \tau}} \leq \min(\sqrt{T}, \frac{1}{\sqrt{2\gamma_3}})$. Thereby,

$$\left\| \frac{\partial}{\partial t} F(g_i) \right\|_{L^2(U)} \leq \min(\sqrt{T}, \frac{1}{\sqrt{2\gamma_3}}) \left\| \frac{\partial g_i}{\partial t} \right\|_{L^\infty(0, t; H^\gamma(U))} + \left\| g_i(0; y) \right\|_{L^2(U)}.$$

In summary,

$$\left\| F(g) \right\|_{\mathbb{H}^{3,1}(0, T; H^\gamma(U))} \leq \frac{1}{\sqrt{2\gamma_1}} \sum_{1 \leq |\beta| \leq \gamma} \left\| D^{\beta-1} \frac{\partial g_i}{\partial t} \right\|_{L^2(0, t; H^\gamma(U))} + \sum_{1 \leq |\beta| \leq \gamma} \left\| D^\beta g_i(0; x) \right\|_{L^2(U)}$$

$$+ \min(\sqrt{T}, \frac{1}{\sqrt{2\gamma_3}}) \left\| \frac{\partial g_i}{\partial t} \right\|_{L^2(0, t; H^\gamma(U))} + \left\| g_i(0; y) \right\|_{L^2(U)}$$

$$\leq \left\| g_i \right\|_{W^{1,2}(0, T; H^\gamma(U))} \sum_{|\beta| \leq \gamma} \max(\frac{1}{\sqrt{2\gamma_1}}, \min(\sqrt{T}, \frac{1}{\sqrt{2\gamma_3}})) + \left\| g_i(0; x) \right\|_{H^\gamma(U)}$$

$$\leq C_1 \left\| g_i \right\|_{W^{1,2}(0, T; H^\gamma(U))} + C_2 \left\| g_i(0; x) \right\|_{H^\gamma(U)}$$

where $C_1 = N^{\gamma+1} \max(\frac{1}{\sqrt{2\gamma_1}}, \min(\sqrt{T}, \frac{1}{\sqrt{2\gamma_3}}))$ and $C_2 = N^{\gamma+1}$. This proves inequality (A.32).

(2). We will show the inequality (A.33) in this part. For the case $|\beta| > 0$. In the same way
for above, we have

\[ \left\| D^\beta F(g_t) \right\|_{L^2(U)} \leq \left\{ \int_U dx \int_0^t d\tau \sum_{i=1}^{\infty} e^{-\lambda_i \tau} D^\beta \varphi_i(x) \left( \varphi_i(\xi), g_t \right) \right\}^{1/2} \]

\[ \leq \left\{ \sum_{i=1}^{\infty} \frac{1}{2\lambda_i} \left\| D^\beta \varphi_i(x) \right\|_{L^2(U)}^2 \int_0^t 2\lambda_i e^{-2\lambda_i \tau} \int_0^t \left( \varphi_i(\xi), g_t \right)^2 \right\}^{1/2} \]

\[ \leq \frac{1}{\sqrt{2\gamma_1}} \left\{ \sum_{i=1}^{\infty} \left\| D^{\beta-1} \varphi_i(x) \right\|_{L^2(U)}^2 \int_0^t 2\lambda_i e^{-2\lambda_i \tau} \int_0^t \left( \varphi_i(\xi), g_t \right)^2 \right\}^{1/2} \]

\[ \leq \frac{1}{\sqrt{2\gamma_1}} \left\| D^{\beta-1} g_t \right\|_{L^2(0,t;L^2(U))} \]

For the case \( \beta = 0 \),

\[ \| F(g_t) \|_{L^2(U)} = \left\{ \int_U dx \int_0^t d\tau \sum_{i=1}^{\infty} e^{-\lambda_i \tau} \varphi_i(x) \left( \varphi_i(\xi), g_t \right) \right\}^{1/2} \]

\[ \leq \min(\sqrt{T}, \frac{1}{\sqrt{2\gamma_3}}) \left\{ \sum_{i=1}^{\infty} \left\| \varphi_i(x) \right\|_{L^2(U)}^2 \int_0^t \left( \varphi_i(\xi), g_t \right)^2 \right\}^{1/2} \]

\[ = \min(\sqrt{T}, \frac{1}{\sqrt{2\gamma_3}}) \| g_t \|_{L^2(0,t;L^2(U))} \]

Thus,

\[ \| F(g) \|_{L^2(0,T;H^{r+2}(U))} \leq C_1 \| g_t \|_{L^2(0,t;H^{\gamma+1}(U))} \]

(3). Now, we shall prove that \( F \) is a compact linear operator.

Let \( g_t \) be a bounded sequence in \( W^{1,2}(0,T;H^{\gamma-1}(U)) \cap L^2(0,T;H^{\gamma+1}(U)) \). Let \( 0 < \beta \leq \gamma - 1 \) when \( \alpha = 1 \) and \( 0 < \beta \leq \gamma + 1 \) when \( \alpha = 0 \). By the Banach-Alaoglu theorem and Canton diagonalization argument, \( g_t \) have a weakly convergence subsequence \( g_{t_1} \) such that (a) \( D^{\beta-1} \frac{\partial^\alpha}{\partial t^\alpha} g_{t_1}(t, x) \) is a weakly convergence subsequence in \( L^2(0,t;L^2(U)) \) and (b) \( D^\beta g_{t_1}(0;x) \) weakly converge in \( L^2(U) \). i.e. for any \( \varepsilon > 0 \), there exists a \( l_\varepsilon \), when \( l_1, l_2 > l_\varepsilon \).

\[ \left\| D^{\beta-1} \frac{\partial^\alpha}{\partial t^\alpha} (g_{t_1} - g_{t_2}) \right\|_{L^2(0,t;L^2(U))} < c_4 \varepsilon \]  \hspace{1cm} (A.37)

and

\[ \left\| D^\beta (g_{t_1} - g_{t_2})(0;x) \right\|_{L^2(U)} < c_4 \varepsilon \]  \hspace{1cm} (A.38)

Here, \( c_4 \) is a constant. Now, denote \( \Delta g_t \) by \( g_{t_1} - g_{t_2} \). The inequality (A.33) follows

\[ \| F(\Delta g) \|_{L^2(0,T;H^{r+2}(U))} \leq C_3 \| \Delta g_t \|_{L^2(0,t;H^{\gamma+1}(U))} \]
The inequality (A.32) implies that
\[ \| F(\Delta g_l) \|_{\mathcal{H}^{3/2}(0,t;H^r(U))} \leq C_1 \| \Delta g_l \|_{W^{1,2}(0,T;H^{r-1/2}(U))} + C_2 \| \Delta g_l(0; x) \|_{H^r(U)}. \]

Above inequalities together with (A.37) and (A.38) indicate that \( F \) is a compact linear operator.

\[ \square \]

### A.2.2 The existence of the global and local solution in \( \mathcal{H}^{s+1/2}_{m+1}(0,T;U) \)

Equation (A.19) is the linearization of the nonlinear parabolic partial difference equation (A.1), which reads
\[ \frac{\partial s_k}{\partial t} + Ls_k = -r_k + \sum_{i=2}^{M} \sum_{n=1}^{N_i} \sum_{j=1}^{i-1} F_{i,j,n}(s_k) \left( \prod_{l=1, l \neq j}^{i-1} F_{i,l,n}(u_k) \right). \quad (A.39) \]

The solution of above equation with respect to the boundary and initial condition (A.45) and (A.46) in \( [0,T] \times U \) can be represented by
\[ s_k(t; x) = \int_0^t \int_U \psi(t; x | \tau; y) [-r_k + \sum_{i=2}^{M} \sum_{n=1}^{N_i} \sum_{j=1}^{i-1} F_{i,j,n}(s_k(\prod_{l=1, l \neq j}^{i-1} F_{i,l,n}(u_k))) dyd\tau \quad (A.40) \]

where \( \psi(t; x | \tau; y) \) is the heat kernel.

Above representation can be simplified to
\[ s_k = A(s_k^b) + f \quad (A.41) \]

where
\[ A(s_k^b) := \int_0^t \int_U \psi(0; x | \tau; y) \sum_{i=2}^{M} \sum_{n=1}^{N_i} \sum_{j=1}^{i-1} F_{i,j,n}(s_k(\prod_{l=1, l \neq j}^{i-1} F_{i,l,n}(u_k))) (t - \tau; y) dyd\tau \quad (A.42) \]

and function \( f \) is defined by
\[ f := -\int_0^t \int_U \psi(0; x | \tau; y) r_k(t - \tau; y) dyd\tau. \quad (A.43) \]

As we described above, equation (A.19) may not have a regular solution \( \{s_k\} \). By the following theorem, we will show that, regardless of \( r_k \), there is a perturbation \( R_k \) of \( r_k \) such that
\[ F_1 s_k = -R_k \]
where
\[
\|R_k - r_k\|_{\mathcal{H}^{m+2s}(0,T;U)} \leq \varepsilon^* \|r_k\|_{\mathcal{H}^{m+2s}(0,T;U)}
\]
for some small \(\varepsilon^*\). The theorem concerning to the solvability of linearized equation follows.

**Theorem 16** Under condition (A1), (A2), and (A3), if \(r_k \in \mathcal{H}^{s+1,2}_{m+1}(0,T;U)\), \(u_k \in \mathcal{H}^{s+1,2}_{m+1}(0,T;U)\), and \(\|u_k\|_{\mathcal{H}^{s+1,2}_{m+2s+1}(0,T;U)} \leq \lambda\), then there exists a function \(R_k \in \mathcal{H}^{s+1,2}_{m+1}(0,T;U)\) satisfying
\[
\|R_k - r_k\|_{\mathcal{H}^{m+2s}(0,T;U)} \leq \varepsilon^* \|r_k\|_{\mathcal{H}^{m+2s}(0,T;U)}
\]
such that the system
\[
\begin{align*}
F_i(s_k) &= -R_k \\
S_k(0,U) &= 0 \\
S_k(t,\partial U) &= 0
\end{align*}
\] (A.44) (A.45) (A.46)
has a solution \(s_k \in \mathcal{H}^{s+1,2}_{m+1}(0,T;U)\). And such a solution obeys
\[
\|s_k\|_{\mathcal{H}^{s+1,2}_{m+2s+1}(0,T;U)} \leq \theta \|r_k\|_{\mathcal{H}^{m+2s}(0,T;U)}
\] (A.47)
where \(\theta\) is independent of \(r_k\) and \(s_k\).

**Proof.** **Step 1:** the existence and regularity of solution.

(1) Define \(X_1\) by \(W^{1,2}(0,T;H^{m+2s}(U)) \cap L^2(0,T;H^{m+2s+2}(U)) \cap \mathcal{S}^0(0,H^{m+2s+1}(U))\) and \(X\) by \(\mathcal{H}^{s+1,2}_{m+2s+1}(0,T;U)\). We assert that \(A(s^k_j)\), which is given by the identity (A.42), is a compact linear operator from \(X_1\) into itself. It suffices to show that
\[
\sum_{i=2}^{M} \sum_{n=1}^{N_i} \sum_{j=1}^{i} F_{i,j,n}(s_k)(\prod_{l=1,l\neq j}^{i-1} F_{i,l,n}(u_k)) \in X_1
\]
because the theorem 15 indicates that \(A(s^k_j)\) is a compact linear operator from \(X_1\) into \(X\). Clearly,
\[
\begin{align*}
\left\| \sum_{i=2}^{M} \sum_{n=1}^{N_i} \sum_{j=1}^{i} F_{i,j,n}(s_k)(\prod_{l=1,l\neq j}^{i-1} F_{i,l,n}(u_k)) \right\|_{W^{1,2}(0,T;H^{m+2s}(U))} \\
\leq \sqrt{T} \left\| \sum_{i=2}^{M} \sum_{n=1}^{N_i} \sum_{j=1}^{i} F_{i,j,n}(s_k)(\prod_{l=1,l\neq j}^{i-1} F_{i,l,n}(u_k)) \right\|_{W^{1,\infty}(0,T;H^{m+2s}(U))}.
\end{align*}
\]
The condition (A3) and the lemma 13 follow
\[
\left\| \sum_{i=2}^{M} \sum_{n=1}^{N_i} \sum_{j=1}^{i-1} F_{i,j,n}(s_k) \left( \prod_{l=1, l \neq j}^{i-1} F_{i,l,n}(u_k) \right) \right\|_{W^{1,\infty}(0,T;H^{m+2s}(U))} \\
\leq \sum_{i=2}^{M} \sum_{n=1}^{N_i} C_i \sum_{j=1}^{i-1} a_{i,j,n} \left\| s_k \right\|_{W^{1,\infty}(0,T;H^{m+2s}(U))} \left( \prod_{l=1, l \neq j}^{i-1} a_{i,l,n} \left\| u_k \right\|_{W^{1,\infty}(0,T;H^{m+2s}(U))} \right)
\]
\[
\leq \sum_{i=2}^{M} \sum_{n=1}^{N_i} C_i \sum_{j=1}^{i-1} a_{i,j,n} \left\| s_k \right\|_{W^{1,\infty}(0,T;H^{m+2s+1}(U))} \left( \prod_{l=1, l \neq j}^{i-1} a_{i,l,n} \left\| u_k \right\|_{W^{1,\infty}(0,T;H^{m+2s+1}(U))} \right)
\]

because \( p_{i,j,n} \leq 1 \). Likewise,
\[
\left\| \sum_{i=2}^{M} \sum_{n=1}^{N_i} \sum_{j=1}^{i-1} F_{i,j,n}(s_k) \left( \prod_{l=1, l \neq j}^{i-1} F_{i,l,n}(u_k) \right) \right\|_{L^2(0,T;H^{m+2s+2}(U))} \\
\leq \sqrt{T} \sum_{i=2}^{M} \sum_{n=1}^{N_i} C_i \sum_{j=1}^{i-1} a_{i,j,n} \left\| s_k \right\|_{L^\infty(0,T;H^{m+2s+3}(U))} \left( \prod_{l=1, l \neq j}^{i-1} a_{i,l,n} \left\| u_k \right\|_{L^\infty(0,T;H^{m+2s+3}(U))} \right)
\]
\[
= \sqrt{T} \sum_{i=2}^{M} \sum_{n=1}^{N_i} C_i \sum_{j=1}^{i-1} a_{i,j,n} \left\| s_k \right\|_{L^0(0,T;H^{m+2s+3}(U))} \left( \prod_{l=1, l \neq j}^{i-1} a_{i,l,n} \left\| u_k \right\|_{L^0(0,T;H^{m+2s+3}(U))} \right).
\]

The initial condition \( s_k \) is zero, i.e. \( s_k(0,x) = 0 \), it turns out that
\[
\left\| \sum_{i=2}^{M} \sum_{n=1}^{N_i} \sum_{j=1}^{i-1} F_{i,j,n}(s_k) \left( \prod_{l=1, l \neq j}^{i-1} F_{i,l,n}(u_k) \right) \right\|_{L^\infty(0,T;H^{m+2s+2}(U))} \\
\leq \sum_{i=2}^{M} \sum_{n=1}^{N_i} C_i \sum_{j=1}^{i-1} a_{i,j,n} \left\| s_k \right\|_{L^\infty(0,T;H^{m+2s+3}(U))} \left( \prod_{l=1, l \neq j}^{i-1} a_{i,l,n} \left\| u_k \right\|_{L^\infty(0,T;H^{m+2s+3}(U))} \right)
\]
\[
= 0.
\]

Above deductions indicate that \( A(s_k^0) \) is a compact linear operator from \( X \) into itself.

The inequality (A.32) and (A.33) immediately turn out that
\[
\left\| f \right\|_{\mathcal{N}^{1,2}_{m+2s+1}(0,T;U)} = \left\| f \right\|_{\mathcal{N}^{1}(0,T;H^{m+2s+1}(U))} + \left\| f \right\|_{\mathcal{N}^{0}(0,T;H^{m+2s+3}(U))} \\
\leq C_1 \left( \left\| r_k \right\|_{W^{1,2}(0,T;H^{m+2s}(U))} + \left\| r_k \right\|_{L^2(0,T;H^{m+2s+2}(U))} \right) + C_2 \left\| r_k(0;x) \right\|_{H^{m+2s+1}(U)} \\
\leq C_1 \left( \left\| r_k \right\|_{W^{1,2}(0,T;H^{m+2s}(U))} + \left\| r_k \right\|_{L^2(0,T;H^{m+2s+2}(U))} \right) + C_2 \left\| r_k(0;x) \right\|_{H^{m+2s+1}(U)} \\
\leq 2\sqrt{T} C_1 \left( \left\| r_k \right\|_{\mathcal{N}^{1,\infty}(0,T;H^{m+2s}(U))} + \left\| r_k \right\|_{\mathcal{N}^{0}(0,T;H^{m+2s+2}(U))} \right) + C_2 \left\| r_k(0;x) \right\|_{H^{m+2s+1}(U)}.
\]

Since \( r_k \in \mathcal{N}^{1,2}_{m+2s}(0,T;U) \), we conclude
\[
\left\| f \right\|_{\mathcal{N}^{1,2}_{m+2s+1}(0,T;U)} \leq C_\gamma \left\| r_k \right\|_{\mathcal{N}^{1,2}_{m+2s}(0,T;U)}.
\] (A.48)
where $C_\eta$ is given by the definition (13). This proves that $f \in X$.

(2). In this part, we will prove that there exist $\eta$ and $s_k \in \mathfrak{H}_{m+2s+1}^{1,2}(0, T; U)$ such that

$$|\eta - 1| \leq \varepsilon_1 < \frac{1}{2} \quad \text{(A.49)}$$

$$\eta s_k^i = A(s_k^i) + f \quad \text{(A.50)}$$

and

$$\|(A - \eta)s_k\|_{\mathfrak{H}_{m+2s+1}^{1,2}} \geq \varepsilon_1 \|s_k\|_{\mathfrak{H}_{m+2s+1}^{1,2}} \quad \text{(A.51)}$$

Since $\mathfrak{H}_{m+2s+1}^{1,2}(0, T; U)$ is a Banach space, so the only limit of eigenvalues of $A$ is zero. By using the Fredholm alternative theorem Gilbarg and Trudinger (2001), we can find out a pair of $\eta_1$ and $\eta_2$, which are in the resolvent of $A$, hold (A.49) and $|\eta_1 - \eta_2| \geq 2\varepsilon_1 - \varepsilon_2$ for any small $\varepsilon_2$ satisfying $0 < \varepsilon_2 \leq \varepsilon_1$. If

$$\|(A - \eta_1)s_k\|_{\mathfrak{H}_{m+2s+1}^{1,2}} \geq \varepsilon_1 \|s_k\|_{\mathfrak{H}_{m+2s+1}^{1,2}} \quad \text{(A.52)}$$

we assign $\eta$ to be $\eta_1$. Otherwise,

$$\|(A - \eta_1)s_k\|_{\mathfrak{H}_{m+2s+1}^{1,2}} < \varepsilon_1 \|s_k\|_{\mathfrak{H}_{m+2s+1}^{1,2}} \quad \text{(A.52)}$$

We compute

$$\|(A - \eta_2)s_k\|_{\mathfrak{H}_{m+2s+1}^{1,2}} \geq \|(\eta_1 - \eta_2)s_k\|_{\mathfrak{H}_{m+2s+1}^{1,2}} - \|(A - \eta_1)s_k\|_{\mathfrak{H}_{m+2s+1}^{1,2}}$$

$$\geq |2\varepsilon_1 - \varepsilon_2 - \varepsilon_1| \|s_k\|_{\mathfrak{H}_{m+2s+1}^{1,2}} \quad \text{(A.52)}$$

Considering that $\varepsilon_2$ is arbitrary, thus, such a $\eta_2$ enables

$$\|(A - \eta_2)s_k\|_{\mathfrak{H}_{m+2s+1}^{1,2}} \geq \varepsilon_1 \|s_k\|_{\mathfrak{H}_{m+2s+1}^{1,2}} \quad \text{(A.53)}$$

Now, we assign $\eta$ to be $\eta_2$. Since $s_k \in \mathfrak{H}_{m+2s+1}^{1,2}(0, T; U)$ and $s, m \geq 1$, $s_k$ is a strong solution. i.e.

$$\frac{\partial s_k}{\partial t} + Ls_k + \frac{1}{\eta} r_k - \frac{1}{\eta} \left( \sum_{i=2}^{M} \sum_{n=1}^{N_i} \sum_{j=1}^{i} F_{i,j,n}(s_k) \prod_{l=1, l \neq j}^{i-1} F_{i,l,n}(u_k) \right) = 0 \quad \text{(A.54)}$$

Let $R_k := r_k - (1 - \eta)(\frac{\partial}{\partial t} + L)s_k$, we have

$$\frac{\partial s_k}{\partial t} = -Ls_k - R_k + \sum_{i=2}^{M} \sum_{n=1}^{N_i} \sum_{j=1}^{i} F_{i,j,n}(s_k) \prod_{l=1, l \neq j}^{i-1} F_{i,l,n}(u_k) \quad \text{(A.55)}$$
By taking a derivative \( \frac{\partial}{\partial t} \) to the both sides of identity (A.55), the right hand side of (A.55) is in \( S_0(0, T; H^{m+2s-1}(U)) \). Thus \( \frac{\partial^2}{\partial t^2} \sigma_k \in S_0(0, T; H^{m+2s-1}(U)) \). This implies that \( \sigma_k \in S^2(0, T; H^{m+2s-1}(U)) \). Taking a derivative \( \frac{\partial}{\partial t} \) to the both sides of identity (A.55) yields \( \sigma_k \in S^3(0, T; H^{m+2(s-1)-1}(U)) \), and so on. By induction, we obtain \( \sigma_k \in S_{m+1,2}^{m+1,2}(0, T; U) \).

**Step 2:** the upper bound of solution.

(3) In this part, we are going to estimate the \( S_{m+2s+1}^{1,2}(0, T; U) \) norm of \( \sigma_k \).

Equation (A.50) reads

\[
\eta s^k_i = A(s^k_j) + f. \tag{A.56}
\]

Hence,

\[
\left\| A(s^k_j) - \eta s^k_i \right\|_{S_{m+2s+1}^{1,2}(0, T; U)} = \left\| f \right\|_{S_{m+2s+1}^{1,2}(0, T; U)}.
\]

Owing to inequality (A.51) and (A.48),

\[
\left\| s^k \right\|_{S_{m+2s+1}^{1,2}(0, T; U)} \leq \frac{C_r}{\varepsilon_1} \left\| r^k \right\|_{S_{m+2s}^{1,2}(0, T; U)}. \tag{A.57}
\]

(4) Since \( R_k := r_k - (1 - \eta)(\frac{\partial}{\partial t} + L)s_k \), we have

\[
\left\| R_k - r_k \right\|_{S_{m+2s}^{1,2}(0, T; U)} \leq \left\| (1 - \eta)\left( \frac{\partial}{\partial t}s_k + Ls_k \right) \right\|_{S_{m+2s}^{1,2}(0, T; U)} \leq \left( \frac{1 - \eta}{\eta} \right) \left( -r_k + \sum_{i=2}^{M} \sum_{n=1}^{N_i} \sum_{j=1}^{i-1} F_{i,j,n}(s_k) \left( \prod_{l=1,l\neq j}^{i-1} F_{i,l,n}(u_k) \right) \right). \]

Considering \( |1 - \eta| \leq \varepsilon_1 < \frac{1}{\eta} \),

\[
\left\| R_k - r_k \right\|_{S_{m+2s}^{1,2}(0, T; U)} \leq \frac{\varepsilon_1}{\eta} \left( \left\| r_k \right\|_{S_{m+2s}^{1,2}(0, T; U)} + \sum_{i=2}^{M} \sum_{n=1}^{N_i} \sum_{j=1}^{i-1} F_{i,j,n}(s_k) \left( \prod_{l=1,l\neq j}^{i-1} F_{i,l,n}(u_k) \right) \right) \]

\[
\leq \frac{\varepsilon_1}{\eta} \left( \left\| r_k \right\|_{S_{m+2s}^{1,2}(0, T; U)} + \sum_{i=2}^{M} \sum_{n=1}^{N_i} C_i^i \sum_{j=1}^{i-1} a_{i,j,n} \left\| s_k \right\|_{S_{m+2s+1}^{1,2}(0, T; U)} \left( \prod_{l=1,l\neq j}^{i-1} a_{i,l,n} \left\| u_k \right\|_{S_{m+2s+1}^{1,2}(0, T; U)} \right) \right) \]

\[
\leq \frac{\varepsilon_1}{\eta} \left( \left\| r_k \right\|_{S_{m+2s}^{1,2}(0, T; U)} + \left\| s_k \right\|_{S_{m+2s+1}^{1,2}(0, T; U)} \right) \left[ M \max(1, C_i^i) \sum_{i=2}^{M} \sum_{n=1}^{N_i} \left( \prod_{l=1}^{i-1} a_{i,l,n} \left\| u_k \right\|_{S_{m+2s+1}^{1,2}(0, T; U)} \right) \right].
\]
Updating the right hand side of above inequality by using the inequality (A.57) and the condition \( \|u_k\|_{N_{m+2}(0,T;U)}^{1,2} \leq \lambda \leq 1 \), it yields

\[
|R_k - r_k|_{N_{m+2}(0,T;U)} \leq \frac{\varepsilon_1}{\eta} \left( \|r_k\|_{N_{m+2}(0,T;U)}^{1,2} \right) \left( 1 + \frac{C_5}{\varepsilon_1} \max(1, C_M^* \left( \sum_{i=2}^{M} \sum_{n=1}^{N_i} \left( \prod_{l=1}^{i} a_{i,l,n} \right) \right)), \right) \leq \frac{\varepsilon_1}{\eta} \|r_k\|_{N_{m+2}(0,T,U)}^{1,2} \left( 1 + \frac{C_5}{\varepsilon_1} \lambda \max(1, C_M^* \left( \sum_{i=2}^{M} \sum_{n=1}^{N_i} \left( \prod_{l=1}^{i} a_{i,l,n} \right) \right)), \right)
\]

Note that \( 0 < \frac{1}{\eta} < 2 \) and

\[
\lambda = \min(1.0, \frac{\varepsilon^*}{4MC_7 \max(1,C_M^*) \sum_{i=2}^{M} \sum_{n=1}^{N_i} \left( \prod_{l=1}^{i} a_{i,l,n} \right) \sum_{i=2}^{M} \sum_{n=1}^{N_i} \left( \prod_{l=1}^{i} a_{i,l,n} \right)})
\]

above inequality is reduced to

\[
\|R_k - r_k\|_{N_{m+2}(0,T;U)} \leq \|r_k\|_{N_{m+2}(0,T;U)}^{1,2} \left( \frac{\varepsilon_1}{\eta} + \varepsilon^* \right).
\]

Recall that \( \varepsilon_1 \) is arbitrary. Assign \( \varepsilon_1 = \varepsilon^*/4 \). We have

\[
\|R_k - r_k\|_{N_{m+2}(0,T;U)} \leq \varepsilon^* \left( \frac{\varepsilon_1}{\eta} \right) \|r_k\|_{N_{m+2}(0,T;U)}^{1,2}.
\]

(A.58)

With this \( \varepsilon_1 \), inequality (A.57) can be rewritten as

\[
\|s_k\|_{N_{m+2}(0,T;U)} \leq \frac{4C_3}{\varepsilon^*} \|r_k\|_{N_{m+2}(0,T;U)}^{1,2} = \theta \|r_k\|_{N_{m+2}(0,T;U)}^{1,2},
\]

where \( \theta \) is function of \( \varepsilon^*, N, U, s, m, T, \) and \( \lambda \). It is easy to check \( R_k \in N_{m+2}(0,T;U) \).

The preceding theorem proves the existence and regularity of the linearized problem of system (A.1).

By the following lemma, we will show the monotonic convergence of \( r_k \) and the boundedness of the nonlinear operator \( F_2(s_k) \) by \( r_k \) in \( N_{m+2}(0,T;U) \).

**Lemma 17** Let

\[
\alpha = C_9 \theta^2 \|r_0\|_{N_{m+2}(0,T;U)}^{1,2}.
\]

Under condition (A1), (A2), and (A3), if i) for every \( 0 < l \leq k \), \( r_l \in N_{m+2}(0,T;U) \) obeys

\[
\|r_l\|_{N_{m+2}(0,T;U)} \leq (1 - \beta) \|r_{l-1}\|_{N_{m+2}(0,T;U)}^{1,2}
\]

(A.60)
where \( r_i \) are given by (A.13).

ii) there exists a function \( R_k \in \mathcal{N}_{m+2s}^{1,2}(0,T;U) \) such that the equation \( F_1(s_k) = -R_k \) has a solution \( s_k \) and

\[
\|s_k\|_{\mathcal{N}_{m+2s}^{1,2}} \leq \theta \|r_k\|_{\mathcal{N}_{m+2s}^{1,2}}(0,T;U) \tag{A.61}
\]

where

\[
\|R_k - r_k\|_{\mathcal{N}_{m+2s}^{1,2}} \leq \varepsilon^* \|r_k\|_{\mathcal{N}_{m+2s}^{1,2}}(0,T;U).
\]

iii) \( u_k \in \mathcal{N}_{m+1}^{1,2}(0,T;U) \) and \( \|u_k\|_{\mathcal{N}_{m+2s+1}^{1,2}}(0,T;U) < \lambda \).

Then there exists a \( \rho_k \) such that

\[
\|F_2(\rho_k s_k)\|_{\mathcal{N}_{m+2s}^{1,2}} \leq \rho_k^2 \alpha \|r_k\|_{\mathcal{N}_{m+2s}^{1,2}}(0,T;U) \tag{A.62}
\]

and

\[
|r_{k+1}|_{\mathcal{N}_{m+2s}^{1,2}} \leq (1 - \beta) \|r_k\|_{\mathcal{N}_{m+2s}^{1,2}}(0,T;U)
\]

where \( i = 1, \ldots N, u_{k+1} = u_k + \rho_k s_k, p^{k+1} = p^k + \rho_k s_{N+1} \).

**Proof.** (1) we shall estimate \( \|F_2(\rho_k s_k)\|_{\mathcal{N}_{m+2s}^{1,2}}(0,T;U) \) in the first part. Considering that \( U \) is a bounded set in \( \mathbb{R}^N \) satisfying a cone condition and \( 2m > N \), by the lemma 13,

\[
\begin{align*}
\|F_2(\rho_k s_k)\|_{\mathcal{N}_{m+2s}^{1,2}}(0,T;U) &= \left\| \sum_{i=2}^{M} \sum_{n=1}^{N_i} \sum_{j=2}^{i} \binom{i}{j} \prod_{l=1}^{j} F_{i-j,l,n}(\rho_k s_k) \prod_{t=1}^{i-j} F_{i_j,l,n}(u_k) \right\|_{\mathcal{N}_{m+2s}^{1,2}}(0,T;U) \\
&\leq \sum_{i=2}^{M} \sum_{n=1}^{N_i} \sum_{j=2}^{i} (C*)^j \binom{i}{j} \prod_{l=1}^{j} \left( \|F_{i-j,l,n}(\rho_k s_k)\|_{\mathcal{N}_{m+2s}^{1,2}}(0,T;U) \right) \prod_{t=1}^{i-j} \left( \|F_{i_j,l,n}(u_k)\|_{\mathcal{N}_{m+2s}^{1,2}}(0,T;U) \right). \\
\end{align*}
\]

Owing to the condition (A3),

\[
\begin{align*}
\|F_2(\rho_k s_k)\|_{\mathcal{N}_{m+2s}^{1,2}}(0,T;U) &\leq \sum_{i=2}^{M} \sum_{n=1}^{N_i} \sum_{j=2}^{i} (C*)^j \binom{i}{j} \prod_{l=1}^{j} (a_{i-l,n} \|\rho_k s_k\|_{\mathcal{N}_{m+2s+1}^{1,2}}(0,T;U)) \prod_{t=1}^{i-j} (a_{i-j,t} \|u_k\|_{\mathcal{N}_{m+2s+1}^{1,2}}(0,T;U)) \\
&\leq 2^M C_*^M \sum_{i=2}^{M} \sum_{n=1}^{N_i} \prod_{l=1}^{i} (a_{i,l,n}) \sum_{j=2}^{i} \|\rho_k s_k\|_{\mathcal{N}_{m+2s+1}^{1,2}}(0,T;U)^2 \|u_k\|_{\mathcal{N}_{m+2s+1}^{1,2}}(0,T;U).
\end{align*}
\]
Since \( \| u_k \|_{\mathbb{S}^{1,2}_{m+2s+1}(0,T;U)} < \lambda \leq 1 \), it turns out that
\[
\| F_2(\rho_k s_k) \|_{\mathbb{S}^{1,2}_{m+2s}(0,T;U)} \leq 2^M C_*^M \sum_{i=2}^{M} \sum_{n=1}^{N_i} \prod_{l=1}^{i-2} \sum_{j=2}^{i} (a_{i,l,n}) \sum_{j=2}^{i} \| \rho_k s_k \|_{\mathbb{S}^{1,2}_{m+2s+1}(0,T;U)}^2.
\]

Due to the inequality (A.61),
\[
\| F_2(\rho_k s_k) \|_{\mathbb{S}^{1,2}_{m+2s}(0,T;U)} \leq 2^M C_*^M \sum_{i=2}^{M} \sum_{n=1}^{N_i} \prod_{l=1}^{i} (a_{i,l,n}) \sum_{j=2}^{i} (\rho_k \theta \| r_k \|_{\mathbb{S}^{1,2}_{m+2s}(0,T;U)})^2.
\]

Note that the inequality (A.60) is satisfied provided \( l \leq k \), thus
\[
\| r_k \|_{\mathbb{S}^{1,2}_{m+2s}(0,T;U)} \leq (1 - \beta)^k \| r_0 \|_{\mathbb{S}^{1,2}_{m+2s}(0,T;U)}. \tag{A.63}
\]

Hence, we compute
\[
\| F_2(\rho_k s_k) \|_{\mathbb{S}^{1,2}_{m+2s}(0,T;U)} \leq [2^M C_*^M \sum_{i=2}^{M} \sum_{n=1}^{N_i} \prod_{l=1}^{i-2} \sum_{j=0}^{i-2} (\rho_k \theta \| r_k \|_{\mathbb{S}^{1,2}_{m+2s}(0,T;U)})^2] \times \rho_k^2 \theta^2 \| r_k \|_{\mathbb{S}^{1,2}_{m+2s}(0,T;U)}.
\]

Let \( \rho_k = \min\left(\frac{2}{3}, \frac{1}{\| r_0 \|_{\mathbb{S}^{1,2}_{m+2s}(0,T;U)}}, \frac{(1-\epsilon^*)(1-\beta)}{2\alpha}\right) \). We have
\[
\| F_2(\rho_k s_k) \|_{\mathbb{S}^{1,2}_{m+2s}(0,T;U)} \leq \rho_k^2 C_0 \theta^2 \| r_k \|_{\mathbb{S}^{1,2}_{m+2s}(0,T;U)} \tag{A.64}
\]

where
\[
C_0 = M 2^M C_*^M \sum_{i=2}^{M} \sum_{n=1}^{N_i} [(i-1) \prod_{l=1}^{i} (a_{i,l,n})].
\]

Due to the fact \( 0 < \beta < 1 \) and the inequality (A.63), it turns out that
\[
\| F_2(\rho_k s_k) \|_{\mathbb{S}^{1,2}_{m+2s}(0,T;U)} \leq \rho_k^2 C_0 \theta^2 (1 - \beta)^k \| r_0 \|_{\mathbb{S}^{1,2}_{m+2s}(0,T;U)} \| r_k \|_{\mathbb{S}^{1,2}_{m+2s}(0,T;U)} \leq \rho_k^2 \alpha \| r_k \|_{\mathbb{S}^{1,2}_{m+2s}(0,T;U)},
\]

where \( \alpha \) is given by (A.59). This proves the inequality (A.62).

(2), By the identity (A.14), \( r_k+1 \) can be decomposed into three parts;
\[
r_{k+1} = r_k + F_1(\rho_k s_k) + F_2(\rho_k s_k).
\]

For simplification, denote \( h_i^k := F_1(s_k) + r_k \). Hence
\[
r_{k+1} = r_k + \rho_k (h_i^k - r_k) + F_2(\rho_k s_k). \tag{A.65}
\]
Taking a $N^{1,2}_{m+2s}(0,T;U)$ norm to both sides of the (A.65), it yields
\[
\|r_{k+1}\|_{N^{1,2}_{m+2s}(0,T;U)} = \|r_{k} + \rho_{k} (h_{k}^{2} - r_{k}) + F_{2}(\rho s_{k})\|_{N^{1,2}_{m+2s}(0,T;U)} \\
\leq |1 - \rho_{k}|\|r_{k}\|_{N^{1,2}_{m+2s}(0,T;U)} + \rho_{k}\|h_{k}^{2}\|_{N^{1,2}_{m+2s}(0,T;U)} + |F_{2}(\rho s_{k})|_{N^{1,2}_{m+2s}(0,T;U)}.
\]
Owing to the inequality (A.62),
\[
|r_{k+1}|_{N^{1,2}_{m+2s}(0,T;U)} \leq (|1 - \rho_{k}| + \rho_{k}^{2}\alpha)\|r_{k}\|_{N^{1,2}_{m+2s}(0,T;U)} + \rho_{k}\|h_{k}^{2}\|_{N^{1,2}_{m+2s}(0,T;U)} \tag{A.66}
\]
Note that $h_{k}^{2} := -R_{k} + r_{k}$, thereby,
\[
\|h_{k}^{2}\|_{N^{1,2}_{m+2s}(0,T;U)} = \|r_{k} - R_{k}\|_{N^{1,2}_{m+2s}(0,T;U)} \leq \varepsilon^{*}\|r_{k}\|_{N^{1,2}_{m+2s}(0,T;U)}.
\]
Inequality (A.66) can be reduced to
\[
\|r_{k+1}\|_{\Omega(0,T;H^{m+2s}(U))} \leq (|1 - \rho_{k}| + \rho_{k}^{2}\alpha + \rho_{k}\varepsilon^{*})\|r_{k}\|_{\Omega(0,T;H^{m+2s}(U))}.
\]
Note that $\rho_{k} = \min(\frac{2}{3}, \frac{1}{\kappa_{0}\|r_{0}\|_{N^{1,2}_{m+2s}(0,T;U)}}(1 - \varepsilon^{*})(1 - \beta))$,\[
|1 - \rho_{k}| + \rho_{k}^{2}\alpha + \rho_{k}\varepsilon^{*} = 1 + \rho_{k}(-1 + \varepsilon^{*} + \rho_{k}\varepsilon^{*}) \leq 1 + \rho_{k}(-1 + \varepsilon^{*} + \frac{(1 - \varepsilon^{*})(1 - \beta)}{2\alpha}) \leq 1 + \rho_{k}(-1 + \varepsilon^{*} + \frac{(1 - \varepsilon^{*})(1 - \beta)}{2}).
\]
Consider that $0 < \beta < 1$ and $\varepsilon^{*} \in [\frac{1}{4}, \frac{1}{2}]$, $\frac{(1 - \varepsilon^{*})}{2} \geq \frac{3}{8}$. Hence,
\[
|1 - \rho_{k}| + \rho_{k}^{2}\alpha + \rho_{k}\varepsilon^{*} \leq 1 - \rho_{k}\left(\frac{1 - \varepsilon^{*}}{2}\right) \leq 1 - \min\left(\frac{1}{4}, \frac{1 - \varepsilon^{*}}{2\kappa_{0}\|r_{0}\|_{N^{1,2}_{m+2s}(0,T;U)}}\right) \leq 1 - \beta.
\]
This proves
\[
\|r_{k+1}\|_{N^{1,2}_{m+2s}(0,T;U)} \leq (1 - \beta)\|r_{k}\|_{N^{1,2}_{m+2s}(0,T;U)}.
\]

The preceding lemma shows the monotonic convergence of the residue sequence. By the following theorem, we will prove the existence of regular solution of the nonlinear parabolic equation (A.1).
Theorem 18 Under condition (A1), (A2), and (A3), if there exists an initial guess \( u_0 \in \mathcal{L}_{m+1,2}^{s+1,2}(0,T;U) \) satisfying the boundary and initial conditions such that \( r_0 \in \mathcal{L}_{m+1,2}^{s+1,2}(0,T;U) \) and

\[
\|u_0\|_{\mathcal{L}_{m+1,2}^{s+1,2}(0,T;U)} + \frac{\theta}{\beta} \|r_0\|_{\mathcal{L}_{m+1,2}^{s+1,2}(0,T;U)} \leq \lambda, \tag{A.67}
\]

then the nonlinear parabolic system (A.1)-(A.2) has a solution \( u^* \) in \( \mathcal{L}_{m+1,2}^{s+1,2}(0,T;U) \).

**Proof.** (1) If \( \|r_0\|_{\mathcal{L}_{m+1,2}^{s+1,2}(0,T;U)} = 0 \), then the initial guess \( u_0 \) is the solution of the nonlinear parabolic system (A.1)-(A.2). This proves the theorem. So, without loss of generality, we assume \( \|r_0\|_{\mathcal{L}_{m+1,2}^{s+1,2}(0,T;U)} > 0 \).

(2) When \( k = 0 \). By theorem 16, there exists a function \( R_0 \) such that

\[
F_1(s_0) = -R_0
\]

has a solution \( s_0 \) in \([0,T] \times U\), where \( s_0 \) and \( R_0 \) hold

\[
\|R_0 - r_0\|_{\mathcal{L}_{m+1,2}^{s+1,2}(0,T;U)} \leq \varepsilon^* \|r_0\|_{\mathcal{L}_{m+1,2}^{s+1,2}(0,T;U)}, \tag{A.68}
\]

\[
\|s_0\|_{\mathcal{L}_{m+1,2}^{s+1,2}(0,T;U)} \leq \theta \|r_0\|_{\mathcal{L}_{m+1,2}^{s+1,2}(0,T;U)}. \tag{A.69}
\]

Denote \( \rho_k \) by

\[
\rho_k = \min\left(\frac{2}{3}, \frac{1}{\theta \|r_0\|_{\mathcal{L}_{m+1,2}^{s+1,2}(0,T;U)}}, \frac{(1 - \varepsilon^*)(1 - \beta)}{2C\theta^2 \|r_0\|_{\mathcal{L}_{m+1,2}^{s+1,2}(0,T;U)}}\right). \tag{A.70}
\]

Let \( u_1 = u_0 + \rho_k s_0 \), we have

\[
\|u_1\|_{\mathcal{L}_{m+1,2}^{s+1,2}(0,T;U)} = \|u_0 + \rho_k s_0\|_{\mathcal{L}_{m+1,2}^{s+1,2}(0,T;U)} \\
\leq \|u_0\|_{\mathcal{L}_{m+1,2}^{s+1,2}(0,T;U)} + \|s_0\|_{\mathcal{L}_{m+1,2}^{s+1,2}(0,T;U)}.
\]

Owing to the inequality (A.69), (A.67), and the fact \( 0 < \beta < 1 \) (identity (A.12)), we have

\[
\|u_1\|_{\mathcal{L}_{m+1,2}^{s+1,2}(0,T;U)} \leq \|u_0\|_{\mathcal{L}_{m+1,2}^{s+1,2}(0,T;U)} + \frac{\theta}{\beta} \|r_0\|_{\mathcal{L}_{m+1,2}^{s+1,2}(0,T;U)}
\leq \lambda.
\]

The identity (A.65) implies that

\[
r_1 = r_0 + \rho_k (h^u_1 - r_0) + F_2(\rho_k s_0).
\]
Taking the $\mathcal{N}^{1,2}_{m+2s}(0,T;U)$ norm to both sides of above identity and taking into account the inequality (A.62) and (A.68) yield

\[
|r^1|_{\mathcal{N}^{1,2}_{m+2s}(0,T;U)} \leq \|r_0\|_{\mathcal{N}^{1,2}_{m+2s}(0,T;U)} \left[1 - \rho_k \left| + \rho_k (\varepsilon^* + \rho_k C_\theta \theta^2 \|r_0\|_{\mathcal{N}^{1,2}_{m+2s}(0,T;U)}) \right] \right].
\]

Furthermore, considering the identity (A.70), we have

\[
1 + \rho_k (-1 + \varepsilon^* + \rho_k C_\theta \theta^2 \|r_0\|_{\mathcal{N}^{1,2}_{m+2s}(0,T;U)}) \\
\leq 1 + \rho_k (-1 + \varepsilon^* + \frac{1 - \varepsilon^*}{2C_\theta \theta^2 \|r_0\|_{\mathcal{N}^{1,2}_{m+2s}(0,T;U)}} C_\theta \theta^2 \|r_0\|_{\mathcal{N}^{1,2}_{m+2s}(0,T;U)}) \\
\leq 1 - \rho_k \left(\frac{1 - \varepsilon^*}{2}\right) \\
\leq 1 - \min \left\{1, \frac{(1 - \varepsilon^*)}{2\theta \|r_0\|_{\mathcal{N}^{1,2}_{m+2s}(0,T;U)}}, \frac{(1 - \varepsilon^*)^2}{4C_\theta \theta^2 \|r_0\|_{\mathcal{N}^{1,2}_{m+2s}(0,T;U)}} \right\}.
\]

This implies that

\[
[1 + \rho_k (-1 + \varepsilon^* + \rho_k C_\theta \|r_0\|_{\mathcal{N}^{1,2}_{m+2s}(0,T;U)})] \leq (1 - \beta). \tag{A.71}
\]

Thus,

\[
\|r_1\|_{\mathcal{N}^{1,2}_{m+2s}(0,T;U)} \leq (1 - \beta) \|r_0\|_{\mathcal{N}^{1,2}_{m+2s}(0,T;U)},
\]

\[
\|u_1\|_{\mathcal{N}^{1,2}_{m+2s+1}(0,T;U)} \leq \lambda,
\]

\[
|s_0|_{\mathcal{N}^{1,2}_{m+2s+1}(0,T;U)} \leq \theta \|r_0\|_{\mathcal{N}^{1,2}_{m+2s}(0,T;U)}.
\]

(2), Without loss of generality, we assume that, for any $1 < \sigma \leq k$,

\[
\|r_\sigma\|_{\mathcal{N}^{1,2}_{m+2s}(0,T;U)} \leq (1 - \beta) \|r_{\sigma-1}\|_{\mathcal{N}^{1,2}_{m+2s}(0,T;U)},
\]

\[
\|u_\sigma\|_{\mathcal{N}^{1,2}_{m+2s+1}(0,T;U)} \leq \lambda,
\]

\[
|s_{\sigma-1}|_{\mathcal{N}^{1,2}_{m+2s+1}(0,T;U)} \leq \theta \|r_{\sigma-1}\|_{\mathcal{N}^{1,2}_{m+2s}(0,T;U)}.
\]

By theorem 16, there exists a function $R_k \in \mathcal{N}^{1,2}_{m+2s}(0,T;U)$ such that the system $F_1(s_k) = -R_k$ has a solution $\{s_k\}$ in $[0,T] \times U$. Such a solution satisfies that

\[
\|R_k - r_k\|_{\mathcal{N}^{1,2}_{m+2s}(0,T;U)} \leq \varepsilon^* \|r_k\|_{\mathcal{N}^{1,2}_{m+2s}(0,T;U)},
\]

\[
\|s_k\|_{\mathcal{N}^{1,2}_{m+2s+1}(0,T;U)} \leq \theta \|r_k\|_{\mathcal{N}^{1,2}_{m+2s}(0,T;U)}.
\]
By lemma 17, we have
\[ \| r_{k+1} \|_{\mathcal{N}_{m+2s+1}^{1,2}(0,T;U)} \leq (1 - \beta) \| r_k \|_{\mathcal{N}_{m+2s}^{1,2}(0,T;U)} . \]

Assign \( u_{k+1} = u_k + \rho_k s_k \). We compute:
\[
\| u_{k+1} \|_{\mathcal{N}_{m+2s+1}^{1,2}(0,T;U)} \leq \| u_k + \rho_k s_k \|_{\mathcal{N}_{m+2s+1}^{1,2}(0,T;U)} \\
\leq \left\| u_0 + \sum_{n=0}^{k} \rho_n s_n \right\|_{\mathcal{N}_{m+2s+1}^{1,2}(0,T;U)} \\
\leq \| u_0 \|_{\mathcal{N}_{m+2s+1}^{1,2}(0,T;U)} + \sum_{n=0}^{k} \| \rho_n s_n \|_{\mathcal{N}_{m+2s+1}^{1,2}(0,T;U)} .
\]

Since \( 0 < \beta, \rho_k < 1 \),
\[
\| u_{k+1} \|_{\mathcal{N}_{m+2s+1}^{1,2}(0,T;U)} \leq \| u_0 \|_{\mathcal{N}_{m+2s+1}^{1,2}(0,T;U)} + \sum_{n=0}^{k} \| s_n \|_{\mathcal{N}_{m+2s+1}^{1,2}(0,T;U)} \\
\leq \| u_0 \|_{\mathcal{N}_{m+2s+1}^{1,2}(0,T;U)} + \theta \sum_{n=0}^{k} \| r_k \|_{\mathcal{N}_{m+2s+1}^{1,2}(0,T;U)} \\
\leq \| u_0 \|_{\mathcal{N}_{m+2s+1}^{1,2}(0,T;U)} + \theta \| r_0 \|_{\mathcal{N}_{m+2s+1}^{1,2}(0,T;U)} \sum_{n=0}^{k} (1 - \beta)^n \\
\leq \lambda .
\]

Likewise, we can prove that \( \frac{\partial u_{k+1}}{\partial t} \) \( u_{k+1} \|_{\mathcal{N}_{m+2s-1}^{1,2}(0,T;U)} \leq \lambda_2 \).

By induction, we conclude that, for any \( k > 0 \),
\[
\| r_{k+1} \|_{\mathcal{N}_{m+2s}^{1,2}(0,T;U)} \leq (1 - \beta) \| r_k \|_{\mathcal{N}_{m+2s}^{1,2}(0,T;U)} ,
\]
\[
\| u_{k+1} \|_{\mathcal{N}_{m+2s+1}^{1,2}(0,T;U)} \leq \lambda ,
\]
\[
\| s_k \|_{\mathcal{N}_{m+2s+1}^{1,2}(0,T;U)} \leq \theta \| r_k \|_{\mathcal{N}_{m+2s}^{1,2}(0,T;U)} .
\]

Considering that \( \beta \) is a nonzero constant independent of \( k \) and \( 0 < \beta < 1 \), sequence \( r_{k+1} \) monotonically converges to 0 in \( \mathcal{N}_{m+2s}^{1,2}(0,T;U) \).

(3). Obviously, the sequence \( u_k \) is Cauchy in the space \( \mathcal{N}_{m+2s+1}^{1,2}(0,T;U) \). This implies that \( u_* \) where \( u_* := \lim_{k \to \infty} u_k \), is the solution of the nonlinear parabolic equation (A.1). Note that \( m > N/2 \) and \( s > 0 \), it is a strong solution in \( \mathcal{N}_{m+1}^{s+1/2}(0,T;U) \).
(4), Clearly, \( u_* \) satisfies the initial and boundary conditions because \( u_0 \) satisfies the initial and boundary conditions and

\[
s_k(0, U) = 0, \quad s_k(t, \partial U) = 0.
\]

Finally, we accomplish the proof. \( \blacksquare \)

The inequality (A.67) is a sufficient condition for the global existence of the strong solution. In the case that the inequality (A.67) fails, we will seek a rescaling of variables such that the inequality (A.67) is held. The rescaling shown by the table A.1 will transfer the system (A.1)-(A.2) into

\[
\frac{\partial \bar{u}}{\partial t} - D_i(\bar{a}^{ij}(x)D_j \bar{u}) + \bar{b} \bar{u} = \bar{f}_0 + \sum_{i=2}^{M} \sum_{n=1}^{N_i} i \bar{F}_{i,j,n}(\bar{u}), \tag{A.72}
\]

\[
\bar{u}(0, U) = \bar{g}(U) = \zeta^\sigma g(U), \quad \bar{u}(t, \partial U) = \bar{h}(t; \partial U) = \zeta^\sigma h(\zeta^\pi - \sigma; \partial U). \tag{A.73}
\]

where \( \zeta \) is a constant, \( \pi, \sigma > 0 \).

<table>
<thead>
<tr>
<th>Ori.</th>
<th>t</th>
<th>x</th>
<th>( u(t; x) )</th>
<th>( f_0 )</th>
<th>( \alpha_{i,j,n} )</th>
<th>( a^{ij} )</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>Res.</td>
<td>( \tilde{t} )</td>
<td>( \tilde{x} )</td>
<td>( \tilde{u}(\tilde{t}; \tilde{x}) )</td>
<td>( \tilde{f}_0 )</td>
<td>( \tilde{a}_{i,j,n} )</td>
<td>( \tilde{a}^{ij} )</td>
<td>( \tilde{b} )</td>
</tr>
<tr>
<td>Relat.</td>
<td>( \tilde{t} = \frac{t}{\zeta^{\sigma-\pi}} )</td>
<td>( \tilde{x} = \frac{x}{\zeta^{\sigma-\pi}} )</td>
<td>( \tilde{u} = \frac{u}{\zeta^{\sigma-\pi}} )</td>
<td>( \tilde{f}_0 = \frac{f_0}{\zeta^{\sigma-\pi}} )</td>
<td>( \tilde{a}<em>{i,j,n} = \frac{\alpha</em>{i,j,n}}{\zeta^{(\sigma-\pi)/\pi}} )</td>
<td>( \tilde{a}^{ij} = \frac{a^{ij}}{\zeta^{\sigma-\pi}} )</td>
<td>( \tilde{b} = \frac{b}{\zeta^{\sigma-\pi}} )</td>
</tr>
</tbody>
</table>

Table A.1 The relationship between the rescaled system and original system.

Note that the rescaling changes the norm but does not change the arguments \( x \). The residual \( \bar{r} \) corresponding to the rescaled system obeys

\[
\bar{r} = \zeta^\sigma r \tag{A.74}
\]

because

\[
\bar{r} : = \frac{\partial \bar{u}}{\partial t} - D_i(\bar{a}^{ij}(x)D_j \bar{u}) + \bar{b} \bar{u} - \bar{f}_0 - \sum_{i=2}^{M} \sum_{n=1}^{N_i} i \bar{F}_{i,j,n}(\bar{u})
\]

\[
= \zeta^\sigma \left[ \frac{\partial u}{\partial t} - D_i(a^{ij}(x)D_j u) + bu - f_0 - \sum_{i=2}^{M} \sum_{n=1}^{N_i} F_{i,j,n}(u) \right]
\]

\[
= \zeta^\sigma r.
\]
For the rescaled system, the condition (A2)-(A3) and the definition 13 are updated to:

(B2), \( \bar{a}^{ij}(x) \in C^{m+2s+1,1}(U) \) and \( b \) is a non-negative constant. \( L \) is self-adjoint, bounded and strong elliptic. i.e. there exist positive number \( \bar{\gamma}_1, \bar{\gamma}_2, \) and a non-negative constant \( \bar{\gamma}_3 \) such that \( \bar{a}^{ij} = \bar{a}^{ji} \),

\[
|a^{ij}(x)\xi_i\xi_j| \geq \bar{\gamma}_1 |\xi|^2, \quad \forall x \in U, \xi \in \mathbb{R}^N. \tag{A.75}
\]

\[
\sum |\bar{a}^{ij}(x)|^2 \leq \bar{\gamma}_2, \quad \forall x \in U. \tag{A.76}
\]

\[
\infty > \bar{b}(x) \geq \bar{\gamma}_3 \geq 0, \quad \forall x \in U. \tag{A.77}
\]

where \( \bar{\gamma}_1 = \zeta^{\sigma-\pi} \bar{\gamma}_1, \bar{\gamma}_2 = \zeta^{\sigma-\pi} \bar{\gamma}_2 \) and \( \bar{\gamma}_3 = \zeta^{\sigma-\pi} \bar{\gamma}_3 \), and \( \bar{a}^{ij} = a^{ij} \). Moreover, \( \bar{g} \in H^{m+2(1+s)}(\bar{U}), \bar{h} \in H^1(0,T;H^{m+2s}(\partial\bar{U})) \cap H^0(0,T;H^{m+2(1+s)}(\partial\bar{U})) \), and \( \bar{f}_0 \in H^{m+2s}(\bar{U}) \) is a force independent of \( \bar{u} \).

(B3), each \( \bar{F}_{i,j} \) is a linear operator from \( H^\alpha(0,T;H^{a+p_i,j,n}(U)) \) into \( H^\alpha(0,T;H^a(U)) \) so that

\[
\left\| \bar{F}_{i,j,n}(\bar{u}) \right\|_{H^\alpha(0,T;H^a(U))} \leq \bar{a}_{i,j,n} \left\| \bar{u} \right\|_{H^\alpha(0,T;H^{a+p_i,j,n}(U))} = \zeta^{(\sigma-i\pi)/i} \alpha_{i,j,n} \left\| \bar{u} \right\|_{H^\alpha(0,T;H^{a+p_i,j,n}(U))}, \tag{A.78}
\]

where \( m + 2s + 1 \geq \alpha \geq 0, s \geq j \geq 0 \) and \( 1 \geq p_i,j,n \geq -\infty \).

**Definition 14** Let the constant \( \varepsilon^* \) to be arbitrary constant in the domain \( [1, 1/2] \), \( \bar{\theta} = \frac{4C_\gamma}{\varepsilon^*} \),

\[
\bar{C}_\gamma = 2N^{m+2(s+1)} \max(1, \frac{\sqrt{T}}{\sqrt{\bar{\gamma}_1}}, \min(T, \frac{\sqrt{T}}{\sqrt{\bar{\gamma}_3}})),
\]

\[
\bar{C}_\theta = M^2 \sum_{i=2}^{M} \sum_{n=1}^{N_i} \prod_{l=1}^{i} (\bar{a}_{i,l}),
\]

\[
\bar{\chi} = \min(1.0, \frac{\varepsilon^*}{4M\bar{C}_\gamma \max(1, \bar{C}_\theta^M) \sum_{i=2}^{M} \sum_{n=1}^{N_i} (\prod_{l=1}^{i} \bar{a}_{i,l,n})}), \tag{A.79}
\]

and

\[
\bar{\beta} = \min\left(\frac{1}{4}, \frac{(1 - \varepsilon^*)}{2\theta \left\| \bar{f}_0 \right\|_{H^{m+2s}(0,T;U)}}, \left(1 - \varepsilon^*\right)^2 \frac{\left\| \bar{f}_0 \right\|_{H^{m+2s}(0,T;U)}}{4C_\theta^2 \bar{\theta} \left\| \bar{f}_0 \right\|_{H^{m+2s}(0,T;U)}}\right). \tag{A.80}
\]
Obviously, the rescaling is identified by the coefficient $\sigma$ and $\pi$. By the following theorem, we will show that regardless of the initial guess $u_0$, we can always find out a pair of $\sigma$, $\pi$, and $T^*$ such that the inequality (A.67) is satisfied.

**Theorem 19** Under condition (A1), (A2), and (A3), if there exists an initial guess $u_0 \in N^{s+1,2}_{m+1}(0,T;U)$ satisfying the boundary and initial conditions, then there is a $T^*$ such that the nonlinear parabolic system (A.1)-(A.2) has a solution $u^*$ in $N^{s+1,2}_{m+1}(0,T^*;U)$, where

\[
T^* = \min(1, \zeta^{M-1} \gamma_1, T),
\]

\[
\zeta = \min\left( \frac{\|u_0\|_{N^{s+1,2}_{m+1}(0,T;U)} + \frac{\theta}{\beta} \|r_0\|_{N^{s+1,2}_{m+2s}(0,T;U)}^{\epsilon}}{\min(1.0, \frac{\epsilon^{\pi}}{4MC_{\gamma, N, i} \sum_{i=2}^{M} \sum_{n=1}^{N_i} \left\|[a_{i,n}]\right\|_{l=1}})}, 1.0, \right),
\]

and the constant $M$ is maximum order of the identity (A.5).

**Proof.** Assign $\sigma = M$ and $\pi = 1$. Define $T^* = \min(1, \zeta^{M-1} \gamma_1, T)$ where $\zeta$ is given by (A.82). It is easy to check that $\overline{C_{\gamma}} \leq C_{\gamma}$ because

\[
\overline{C_{\gamma}} = 2N^{m+2(s+1)} \max(1, \frac{\sqrt{T^*}}{\sqrt{\gamma_1}}, \min(T^*, \frac{\sqrt{T^*}}{\sqrt{\gamma_3}})) = 2N^{m+2(s+1)} \leq C_{\gamma}.
\]

Furthermore, $\overline{\theta} \leq \theta$. Considering that $0 < \zeta \leq 1$, we have

\[
\lambda = \min(1.0, \frac{\epsilon^{\pi}}{4MC_{\gamma, N, i} \sum_{i=2}^{M} \sum_{n=1}^{N_i} \left\|[a_{i,n}]\right\|_{l=1}});
\]

\[
\geq \min(1.0, \frac{\epsilon^{\pi}}{4MC_{\gamma, N, i} \sum_{i=2}^{M} \sum_{n=1}^{N_i} \left\|[a_{i,n}]\right\|_{l=1}}) = \lambda(T^*)
\]

and

\[
\frac{1}{\beta} = \max(4, \frac{2\theta \|r_0\|_{N^{s+1,2}_{m+2s}(0,T;U)}}{(1 - \epsilon^{\pi})}, \frac{4C_{\theta} \theta^2 \|r_0\|_{N^{s+1,2}_{m+2s}(0,T;U)}^{\epsilon}}{(1 - \epsilon^{\pi})^2} \leq \max(4, \frac{2\theta \|r_0\|_{N^{s+1,2}_{m+2s}(0,T;U)}^{\epsilon}}{(1 - \epsilon^{\pi})}, \frac{4C_{\theta} \theta^2 \|r_0\|_{N^{s+1,2}_{m+2s}(0,T;U)}^{\epsilon}}{(1 - \epsilon^{\pi})^2}) \leq \frac{1}{\beta}.
\]
Note that $C_*$ is independent of $T$ which is proved by lemma 13. Therefore, by the identity (A.82) and (A.79):

$$
\|\bar{u}_0\|_{\mathcal{N}_{m+2s+1}^{s+1,2}(0,T^*;U)} + \frac{\bar{\theta}}{\beta} \|\bar{r}_0\|_{\mathcal{N}_{m+2s}^{s+1,2}(0,T^*;U)} \\
\leq \|\bar{u}_0\|_{\mathcal{N}_{m+2s+1}^{s+1,2}(0,T;U)} + \frac{\theta}{\beta} \|\bar{r}_0\|_{\mathcal{N}_{m+2s}^{s+1,2}(0,T;U)} \\
\leq \zeta(\|u_0\|_{\mathcal{N}_{m+2s+1}^{s+1,2}(0,T^*;U)} + \frac{\theta}{\beta} \|r_0\|_{\mathcal{N}_{m+2s}^{s+1,2}(0,T^*;U)} S^{M-1}) \\
\leq \min(1,\epsilon^* \frac{\epsilon^*}{4MC_\gamma \max(1, C_*^M) \sum_{i=2}^{M} \sum_{n=1}^{N_i} \prod_{l=1}^{i} a_{i,l,n}}) \\
\leq \bar{\lambda}.
$$

Hence, by theorem 18, we prove the existence of the strong solution in $\mathcal{N}_{m+1}^{s+1,2}(0,T^*;U)$. ■

Above theorem prove the local existence of the Dirichlet problem of nonlinear system (A.1)-(A.2).

### A.3 Conclusion

This paper proposes a monotonic method to approach the strong solutions of nonlinear parabolic system (A.1)-(A.2). By this method, we prove that if the initial data satisfies inequality (A.67), then the global solution of system (A.1)-(A.2) exists in $\mathcal{N}_{m+1}^{s+1,2}(0,T;U)$. Otherwise, there exists a $T^*$ such that the solution exists in $\mathcal{N}_{m+2s+2}^{s+1,2}(0,T^*;U)$. And theorem 19 gives a time interval that the solution will not blow up.
BIBLIOGRAPHY


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