NUMERICAL ANALYSIS QUALIFYING EXAM

Spring 2008
Thursday, January 10 2008, 9:00 am - 1:00 pm
Room 305 Carver

Instructions

- Write your complete ISU ID number on every page that you turn in. Do NOT write your name on any sheet that you turn in.
- Work all 6 problems. Start each problem on a separate sheet of paper, and clearly indicate the problem number.
- Every effort is made to proofread the exam, but misprints may occur. If you believe that a problem has been stated incorrectly, check with the proctor and indicate your interpretation in the solution. Do not interpret the problem in a way that it becomes trivial.

1. Let A and B be matrices in $\mathbb{R}^{n\times n}$, A be non-singular, and satisfy the inequality $\|A^{-1}\|_2\|B\|_2 = q < 1$. Here $\|\cdot\|_2$ is the matrix norm subordinate to the Euclidean norm in $\mathbb{R}^n$.

(a) Show that $C = A + B$ is non-singular.

(b) Show that the iteration process $Ax^{j+1} = b - Bx^j$, $j = 0, 1, \cdots$ converges for any x^0 to the solution of the system $Cx = b$. Give an estimate for the Euclidean norm of error $x^j - x$ in terms of q.

2. An $n \times n$ matrix of the form $N(\mathbf{y}, k) = I - \mathbf{y}\mathbf{e}_k^T$ is called a *Gauss-Jordan matrix.* Here $\mathbf{e}_k$ is the kth unit coordinate vector, and $\mathbf{y}$ is an arbitrary vector.

(a) Find a formula for $N(\mathbf{y}, k)^{-1}$. Under what conditions does this inverse exist?

(b) Let $\mathbf{x}$ be an arbitrary vector. For given k, find a vector $\mathbf{y}$ so that $N(\mathbf{y}, k)\mathbf{x} = \mathbf{e}_k$. Under what conditions does such a $\mathbf{y}$ exist?

(c) For a given square matrix A, give an algorithm based on Gauss-Jordan matrices that computes A^{-1}. Under what conditions will this algorithm work?

3. Let $f(x)$ be a twice continuously differentiable real-valued function defined on $\mathbb{R}$. Suppose that: i) $f(x^*) = 0$, ii) $f'(x^*) = 0$, and $f''(x^*) \neq 0$. Assuming that $f''(x)$ is Lipschitz continuous on $\mathbb{R}$, prove that the there is a interval $I(\delta) = \{x \in \mathbb{R} : \ |x - x^*| \leq \delta\}$ such that the iteration scheme

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n)}, \qquad x_0 \in I(\delta)$$

converges quadratically to x^*.

4. Let $f \in C[a, 2b-a]$, and consider the nonsymmetric quadrature rule

$$\int_a^b f(x)dx = \frac{b-a}{12}\left[5f(a) + 8f(b) - f(2b-a)\right] + Rf.$$

(a) Show $Rp = 0$ for all $p \in P_2$ (polynomials of degree ≤ 2).
(b) Assume $f \in C^3[a, 2b-a]$, prove the following error formula:

$$Rf = \frac{(b-a)^4}{24} f'''(\xi), \quad \xi \in [a, 2b-a].$$

5. Consider the method

$$y_{n+2} - y_{n+1} = \frac{h}{12}[4f(x_{n+2}, y_{n+2}) + 8f(x_{n+1}, y_{n+1}) - f(x_n, y_n)]$$

for approximating solutions of $y' = f(x, y)$, $y(0) = y_0$. Here $x_n = nh$.

(a) Determine the order of the method.
(b) Apply the method to the scalar initial value problem $y' = x$, $y(0) = 0$ to get a one-step method of the form $y_{n+2} - y_{n+1} = \phi(n, h)$. Find the exact solution of this difference equation satisfying the initial condition $y_1 = h^2/2 (= x_1^2/2)$. **Note:** A particular solution of the difference equation can be found in the form $y_n = An^2 + Bn$.
(c) Compute the fixed station limit

$$\lim_{h \to 0, x = nh} y_n$$

of the solution you found
(d) Is the method convergent in this case? Justify your answer and indicate what properties the method has or does not have that can help explain these results.

6. Let $x_0, \cdots, x_n$ be the zeros of the monic Legendre polynomial L_{n+1}. Show:
(a) For every polynomial $p \in P_n$, the set of all polynomials of degree n,

$$\|p\|_2 \leq \sqrt{2} \max_{0 \leq k \leq n} |p(x_k)|.$$

(b)Let $f \in C[-1, +1]$ and suppose that , for each n, $\tilde{p}_n$ is the polynomial that interpolates f at $x_0, \cdots, x_n$. Then

$$\lim_{n \to \infty} \|f - \tilde{p}_n\|_2 = 0.$$