NUMERICAL ANALYSIS QUALIFYING EXAM

Spring 2007
Saturday, January 6, 2007, 9:00 am - 12:00 noon
Room 305 Carver

Instructions:

- Write your complete student identification number on every page that you turn in. **DO NOT** write your name on any sheet that you turn in.
- Work all 6 problems. Start each problem on a separate sheet of paper, and clearly indicate the problem number.

(1) The least squares solution to the linear system $Ax = b$ is the vector x which minimizes $\|Ax - b\|_2$. If there is more than one such x, we pick the one that minimizes $\|x\|_2$.

(a) Find the least squares solution to

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1.1 \end{bmatrix} x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

(b) Find the least squares solution to

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

(c) By using the results of (a),(b) determine a lower bound on the condition number, in the ∞-norm, of problem (b) with respect to changes in the matrix A.

(2) Let

$$A = \begin{bmatrix} a & -b \\ -a & a \end{bmatrix}, \qquad a > b > 0.$$

Show that Gauss-Seidel iteration, for solving $Ax = b$, converges for this choice of A.

(3) Let $A \in \mathbb{R}^{m\times m}$ be symmetric positive definite, $b \in \mathbb{R}^m$ and define $\phi : \mathbb{R}^m \to \mathbb{R}$ by

$$\phi(x) = \tfrac{1}{2}x^T Ax - x^T b.$$

Suppose that K is a subspace of $\mathbb{R}^m$. Show that $\hat{x} \in K$ minimizes $\phi(x)$ over K if and only if $\nabla\phi(\hat{x}) \perp K$.

(4) The Chebyshev polynomial of degree n, $T_n(x)$, is defined on $[-1, 1]$ by $T_n(\cos\theta) = \cos(n\theta)$, with $x = \cos\theta$.

(a) Show the sequence of polynomials $\{T_n(x)\}_{n=0}^{\infty}$ satisfies the 3-term recurrence relation

$$T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x).$$

(b) Suppose that $f(x) = \ln(2 + x)$, $x \in [-1, 1]$, is interpolated by a polynomial of degree n at the Chebyshev points (i.e. the zeros of $T_{n+1}(x)$) $x_k = \cos((2k + 1)\pi/(2n + 2))$, $k = 0, 1, \ldots, n$. Derive a bound on the maximum error

$$\|f - p_n\|_\infty = \max_{|x|\le 1} |f(x) - p_n(x)|$$

that depends only on the integer n.

(5) Show that the 1-stage implicit Runge-Kutta method

$$y_{n+1} = y_n + hk_1, \qquad k_1 = f(t_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}hk_1).$$

is a method of order 2 and determine the principal error function in terms of f and its first and second order derivatives. To simplify notation you should write f, f_t, f_y, etc. rather than $f(t_n, y_n)$, $f_t(t_n, y_n)$, $f_y(t_n, y_n)$. Also assume the problem is a scalar problem.

(6) (a) Let $g(t)$ be a continuous function defined on $[-1, 1]$, and let $p_2(t)$ be the interpolating polynomial for $g(t)$ at $-1, 0, 1$. Derive Simpson's rule for the integral

$$\int_{-1}^{1} g(t)\, dt$$

by integrating $p_2(t)$ over this interval.

(b) Let $r_2(t) = g(t) - p_2(t)$. Show that the corresponding error in Simpson's rule is

$$\int_{-1}^{1} r_2(t)\, dt = -\frac{1}{90} g^{(4)}(\eta), \quad \text{for some } \eta \in (-1, 1).$$