NUMERICAL ANALYSIS QUALIFYING EXAM

Spring 2006
Saturday, 2006, 9:00 am - 12:00 noon
Room 305 Carver

Instructions

- Write your complete social security number on every page that you turn in. Do NOT write your name on any sheet that you turn in.
- Work all 6 problems. Start each problem on a separate sheet of paper, and clearly indicate the problem number.

1. Let A be a Hermitian, positive definite $n \times n$ matrix. After k steps of Gaussian elimination without pivoting, A will be reduced to the form

$$A^{(k)} = \begin{pmatrix} A_{11}^{(k)} & A_{12}^{(k)} \\ 0 & A_{22}^{(k)} \end{pmatrix}$$

where $A_{22}^{(k)}$ is an $(n-k) \times (n-k)$ matrix. Show by induction that

(a) $A_{22}^{(k)}$ is symmetric and positive definite

(b) $a_{ii}^{(k)} \leq a_{ii}^{(k-1)}$ for $k \leq i \leq n$, $k = 1, 2, \ldots, n-1$.

2. Consider the numerical solution of the system of linear equations

$$Ax = b$$

with

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

(a) Find the condition number κ of A in the ∞-norm.

(b) Find a right-hand side b and perturbation Δb for which this condition number is actually achieved.

(c) For the same b, find a perturbation ΔA for which this condition number is approximately achieved.

3. Consider the two-step Newton method

$$y^{(k)} = x^{(k)} - \frac{f(x^{(k)})}{f'(x^{(k)})}, \quad x^{(k+1)} = y^{(k)} - \frac{f(y^{(k)})}{f'(x^{(k)})}$$

for the solution of the equation $f(x) = 0$, and show:

(a) If the method converges, then

$$\lim_{k\to\infty} \frac{(x^{(k+1)} - \xi)}{(y^{(k)} - \xi)(x^{(k)} - \xi)} = \frac{f''(\xi)}{f'(\xi)},$$

where ξ is the solution.

(b) The convergence is cubic:

$$\lim_{k\to\infty} \frac{(x^{(k+1)} - \xi)}{(x^{(k)} - \xi)^3} = \frac{1}{2}\left[\frac{f''(\xi)}{f'(\xi)}\right]^2.$$

4. Find an explicit expression for the polynomial of degree 20 that satisfies $p(0) = 2$, $p'(0) = 0$, $p(1) = p'(1) = p^{(2)}(1) = \cdots = p^{(16)}(1) = 0$, $p^{(17)}(1) = 5$. (Hint: do not attempt to solve a large linear system of equations by hand; construct two suitable basis polynomials.)

5. Let p be a polynomial of degree $\leq n-1$ that interpolates the function $f(x) = \sinh x$ at any set of n nodes in the interval $[-1, 1]$, subject only to the condition that one of the nodes is 0. Prove that the error obeys this inequality on $[-1, 1]$:

$$|p(x) - f(x)| \leq \frac{2^n}{n!}|f(x)|.$$

6. The two-step Adams-Bashforth method for the numerical solution of

$$y'(t) = f(t, y(t))$$

is given by

$$y_{n+1} = y_n + \frac{h}{2}\left[3f(t_n, y_n) - f(t_{n-1}, y_{n-1})\right],$$

where

$$t_n = t_0 + nh.$$

One of the difficulties with multistep methods such as this is the change of stepsize. For the case

$$t_n = t_{n-1} + h,$$
$$t_{n+1} = t_n + \alpha h$$

(changing the step size from h to αh), derive a corresponding formula. That is, find β and γ so that the method

$$y_{n+1} = y_n + h\left[\beta f(t_n, y_n) + \gamma f(t_{n-1}, y_{n-1})\right]$$

is of maximal order.