
NUMERICAL ANALYSIS WRITTEN QUALIFYING EXAM
January 12, 2002

Instructions:

- Put your Social Security Number on each page of your exam. Do not put your name on the exam.
- Solve a total of 6 problems as completely as possible. These 6 problems should be chosen using the following criteria:
  - · choose 2 problems from Part I (linear algebra)
  - · choose 1 problem from Part II (nonlinear equations)
  - · choose 2 problems from Part III (approximation theory, interpolation and numerical integration)
  - · choose 1 problem from Part IV (numerical ODEs)
- Only 6 problems will be graded. If you attempt more than 6 problems, indicate which 6 are to be graded. If you fail to do this, your first 6 problems will be graded.

PART I - Linear Algebra
Complete 2 problems

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1. *QR decomposition*

Let $\mathbf{u}, \mathbf{v} \in \mathbf{R}^m$ and let $\sigma \in \mathbf{R}^1$. Define

$$H(\mathbf{u}, \mathbf{v}, \sigma) = I - \sigma \mathbf{u}\mathbf{v}^T ,$$

where I represents the $m \times m$ identity matrix.

a. Determine σ such that $H(\mathbf{u}, \mathbf{u}, \sigma)$ is orthogonal.

b. Let $\mathbf{x} \in \mathbf{R}^m$, $\mathbf{x} \neq 0$. Show how to choose a vector $\mathbf{u}$ such that $H = H(\mathbf{u}, \mathbf{u}, \sigma)$ has the property that $H\mathbf{x}$ is a multiple of $\mathbf{e}^{(1)} = (1, 0, 0, \ldots, 0)^T$ where σ is defined in (a.).

c. Let A be a real $m \times n$ matrix. From (b) we know that we can construct transformations $H^{(k)}$. $k = 1. \ldots, \ell$, $\ell \leq \min(m-1, n)$ such that

$$A^{(\ell+1)} = H^{(\ell)} H^{(\ell-1)} \cdots H^{(1)} A$$

has row echelon structure. Use this result to state and prove the existence of a QR decomposition of a real $m \times n$ matrix A.

d. Prove or give a counterexample:

> The QR decompositon of an arbitrary $m \times n$ matrix is unique.

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2. *Iterative methods for linear systems*

Suppose we have the iterative method

(*) $$\mathbf{x}^{k+1} = H\mathbf{x}^k + \mathbf{d}$$

for solving the linear system $A\mathbf{x} = \mathbf{b}$ where A, H are $n \times n$ matrices and $\mathbf{x}, \mathbf{b}, \mathbf{d} \in \mathbf{R}^n$.

a. Assume that the equation $\mathbf{x} = H\mathbf{x} + \mathbf{d}$ has a unique solution $\mathbf{x}^*$. State (don't prove) a necessary and sufficient condition for the iterates defined by (*) above to converge to x^* for any $\mathbf{x}^0$.

b. The SOR iteration is obtained by taking a weighted average of the last iterate and the Gauss-Seidel iterate. Take ω to be the parameter and write the SOR iteration in the form $\mathbf{x}^{k+1} = H_\omega \mathbf{x}^k + \mathbf{d}$ and explicitly give H_ω and $\mathbf{d}$ in terms of the original splitting of A.

c. We want to prove the following result:

Let A be an $n \times n$ matrix which has nonzero diagonal elements and let H_ω be the iteration matrix in (b) for the SOR iteration. Then

$$\rho(H_\omega) \geq |\omega - 1| .$$

(i) To prove this, first show that

$$\text{determinant of } H_\omega = (1-\omega)^n .$$

(ii) Write the determinant of H_ω in terms of the eigenvalues of H_ω and use this to conclude the desired result.

d. What does the result in (c) say about the choice of the parameter ω in the SOR iteration?

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3. *Eigenvalue Approximations*

Recall the *power method* for finding a dominant eigenvalue and associated eigenvector of an $n \times n$ real matrix A.

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choose a unit 2-norm vector $\mathbf{q}^{(0)}$
for $k = 1, 2, \ldots$
    $\mathbf{z}^{(k)} = A\mathbf{q}^{(k-1)}$
    $\mathbf{q}^{(k)} = \mathbf{z}^{(k)}/||\mathbf{z}^{(k)}||_2$
    $\lambda^{(k)} = (\mathbf{q}^{(k)})^T A\mathbf{q}^{(k)}$
end
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a. Under what assumptions on A will this method converge to a dominant eigenvalue, λ_1, of A? What happens if these assumptions do not hold?

b. Under the assumptions of part (a), show that

$$|\lambda_1 - \lambda^{(k)}| = O\left(\left| \frac{\lambda_2}{\lambda_1} \right|^k \right) ,$$

where λ_2 is the second dominant eigenvalue of A.

c. Describe the *inverse iteration method* using a shift σ.

PART II - Nonlinear Equations
Complete 1 problem

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4. Consider the following nonlinear system $F(x) = 0$:

$$\begin{aligned} x_1^2 - 81(x_2 + 0.1)^2 + \sin x_3 + 1.06 &= 0 \\ 3x_1 - \cos(x_2 x_3) - 0.5 &= 0 \\ e^{x_1 x_2} + 20x_3 + (10\pi - 3)/3 &= 0\,. \end{aligned}$$

a. Put $F(x) = 0$ into a form $x = G(x)$ and show, using the contraction mapping theorem, that the system has a unique solution in

$$D = \{x = (x_1, x_2, x_3) \mid -1 \le x_i \le 1,\ i = 1, 2, 3\}\,.$$

b. Compute the Jacobian matrix $J(x)$ for the system $F(x) = 0$.

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5. Let $G : D \subset \mathbb{R}^n \to \mathbb{R}^n$.

a. What property must G satisfy in order to be a contraction on the set D?

b. If G is defined by $Gx = Ax + d$ where A is an $n \times n$ matrix and $d \in \mathbb{R}^n$, show that there is a norm on $\mathbb{R}^n$ in which G is a contraction if and only if $\rho(A) < 1$ where $\rho(A)$ denote the spectral radius of A.

c. Assume that G is a contraction on the convex set D with contraction constant α. Further assume that there exists $x^0 \in D$ such that

$$S = \left\{ x \mid \|x - Gx^0\| \le \frac{\alpha}{1-\alpha} \|Gx^0 - x^0\| \right\} \subset D\,.$$

Show that the fixed point iteration $x^{k+1} = Gx^k$ converges to a unique fixed point x^* of G in S.

PART III - Approximation Theory, Interpolation & Numerical Integration
Complete 2 problems

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6. Let V denote an inner product space with the inner product denoted by $(\cdot,\cdot)$. Let $\{\phi_1, \phi_2, \ldots, \phi_n\}$ be a set of orthonormal vectors in V and let $V_n = \text{span}\{\phi_1, \phi_2, \ldots, \phi_n\}$. Let $u \in V$ be a given vector.

a. Determine the best approximation to $u \in V$ out of V_n with respect to the given inner product.

b. Suppose a vector $\psi \in V$ is added to the set $\{\phi_1, \phi_2, \ldots, \phi_n\}$ to form the new set $\{\phi_1, \phi_2, \ldots, \phi_n, \psi\}$. Suppose that the set $\{\phi_1, \phi_2, \ldots, \phi_n, \psi\}$ is linearly dependent so that ψ may be written as a linear combination of the vectors in $\{\phi_1, \phi_2, \ldots, \phi_n\}$. Let $W = \text{span}\{\phi_1, \phi_2, \ldots, \phi_n, \psi\}$. Prove that the best approximation to $u \in V$ out of W with respect to the given inner product is the same as that obtained in part (a).

c. Suppose you do not know that the set $\{\phi_1, \phi_2, \ldots, \phi_n, \psi\}$ defined in part (b) is linearly dependent. However, you do know that ψ is not orthogonal to the other elements $\phi_1, \phi_2, \ldots, \phi_n$ of the spanning set for W. You then try to solve for the best approximation to $u \in V$ out of W on a computer. Lo and behold, you find that the matrix you have to invert is singular. Prove this fact.

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7. Suppose $H(x)$ is a piecewise cubic polynomial interpolating a function $f(x)$ as follows:

$$H(\xi_i) = f(\xi_i), \quad H'(\xi_i) = f'(\xi_i), \quad i = 0, 1, \ldots, N\,,$$

where ξ_i's form a partition of $[a,b]$ such that

$$\begin{aligned} a &= \xi_0 < \xi_1 < \cdots < \xi_N = b \\ h &= \xi_i - \xi_{i-1}, \quad i = 1, 2, \ldots, N\,. \end{aligned}$$

Define $R(f;x)$ to be the error function given by

$$R(f;x) = f(x) - H(x)\,,$$

and assume that $f(x)$ is in $C^4([a,b])$.

a. Show that

$$\frac{d^4}{dx^4}R(f;x) = \frac{d^4}{dx^4}f(x)\,.$$

b. Show that for $x \in [\xi_i, \xi_{i+1}]$, there exists a $y \in (\xi_i, \xi_{i+1})$ such that

$$R(f;x) = \frac{(x - x_i)^2(x - x_{i+1})^2}{4!}f^4(y)\,.$$

c. Show that

$$\max_{a \le x \le b} |R(f;x)| \le Ch^4\,.$$

NUMERICAL INTEGRATION

8. Euler's constant is defined by

$$\gamma = \lim_{k\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} - \ln k\right) = .577215\ldots$$

a) Why would the computation of γ from this definition be very inaccurate?

b) Fortunately, γ also has *integral* representations. Consider the formulas:

$$(i) \quad \gamma = -\int_0^\infty e^{-x}\ln x \; dx$$

$$(ii) \quad \gamma = \int_0^\infty \left((1+x)^{-1} - e^{-x}\right) x^{-1} \; dx$$

$$(iii) \quad \gamma = \int_0^1 \ln(-\ln x) \; dx \, .$$

Describe how you would approach the problem with each of the representations (i)–(iii) above and discuss the advantages and disadvantages of each.

PART IV - Numerical ODEs
Complete 1 problem

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9. For the initial value problem

$$y' = f(t, y), \quad y(t_0) = y_0$$

on the interval $[t_0, T]$, consider the scheme

$$\begin{cases} Y_0 = y_0, \quad Y_1 = Y_0 + hf(t_1, Y_0) \\ Y_{n+1} = Y_{n-1} + 2hf(t_n, Y_n), \qquad n = 1, \cdots, N-1 \end{cases}$$

where $h = (T - t_0)/N$ and $t_n = t_0 + nh$.

a. Check the consistency and stability of the scheme.

b. Is the scheme convergent? What is the order of accuracy of the scheme?

c. If $f(t, y) = -y$ and denote $e_n = y(t_n) - Y_n$, prove that $e_{n+1} + 2he_n - e_{n-1} = O(h^3)$.

d. If you neglect the error $O(h^3)$ in part c, i.e., you assume that $e_{n+1} + 2he_n - e_{n-1} = 0$, prove that $e_n = O(h^2)$. (For part d, do not use the Equivalence Theorem.)

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10. Consider the initial value problem

$$y'' = f(t, y, y'), \quad y(t_0) = y_0, \; y'(t_0) = z_0$$

on the interval $[t_0, T]$.

a. Convert this initial value problem into one for a system of first order ODEs.

b. Define the backward Euler scheme for the system in a.

c. Without using the Equivalence Theorem, prove the convergence of the backward Euler scheme in part b