

**Numerical Analysis Qualifying Exam**  
August 22, 2009

**Instructions**

- Write your ISU ID Number on every page that you turn in. Do not write your name on any sheet that you turn in
- Work all 6 problems. Start each problem on a separate sheet of paper and clearly indicate the problem number

1. (a) Show that if  $0 \neq \mathbf{s} \in \mathbf{R}^n$  and  $E \in \mathbf{R}^{n \times n}$ , then  $\|E(I - \mathbf{s}\mathbf{s}^T / \mathbf{s}^T \mathbf{s})\|_F^2 = \|E\|_F^2 - \|E\mathbf{s}\|_2^2 / \mathbf{s}^T \mathbf{s}$ .  
(b) Suppose  $\mathbf{u} \in \mathbf{R}^m$  and  $\mathbf{v} \in \mathbf{R}^n$ . Show that if  $E = \mathbf{u}\mathbf{v}^T$ , then  $\|E\|_F = \|E\|_2 = \|\mathbf{u}\|_2 \|\mathbf{v}\|_2$ .  
(c) Suppose  $A \in \mathbf{R}^{m \times n}$ ,  $\mathbf{y} \in \mathbf{R}^m$ , and  $0 \neq \mathbf{s} \in \mathbf{R}^n$ . Show that  $E = (\mathbf{y} - A\mathbf{s})\mathbf{s}^T / \mathbf{s}^T \mathbf{s}$  has the smallest 2-norm of all  $m$  by  $n$  matrices  $E$  that satisfy  $(A + E)\mathbf{s} = \mathbf{y}$ .
2. (a) A set of nonzero vectors  $\{\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_{n-1}\}$  in  $\mathbf{R}^n$  is said to be conjugate with respect to a symmetric positive definite matrix  $A$  if  $\mathbf{p}_i^T A \mathbf{p}_j = 0$ ,  $\forall i \neq j$ . Show that  $\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_{n-1}$  are linearly independent.  
(b) Consider  $A\mathbf{x} = \mathbf{b}$ . Let  $\{\mathbf{x}_k\}$  be a sequence in  $\mathbf{R}^n$ ,  $\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k \mathbf{p}_k$ , where  $\alpha_k = \mathbf{p}_k^T (\mathbf{b} - A\mathbf{x}_k) / \mathbf{p}_k^T A \mathbf{p}_k$ . Show that  $\mathbf{x}_n$  solves the equation  $A\mathbf{x} = \mathbf{b}$ .
3. Let  $n$  be a positive integer and  $f \in C^{n+1}[a, b]$  (i.e.,  $f$  is a  $(n+1)$  times differentiable function). Let  $p_n(x)$  denote the Lagrange interpolation polynomial of degree  $n$  with interpolation nodes  $x_0, x_1, \dots, x_n \in [a, b]$ .
  - a) Prove that the Lagrange interpolation polynomial is unique.
  - b) Give an explicit formula for the Lagrange interpolation polynomial  $p_n(x)$ .
  - c) Prove the following error formula:

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^n (x - x_i)$$

for some  $\xi \in (a, b)$ . (Hint: consider the number of zeros of the function  $g(t) = f(t) - p_n(t) - [f(x) - p_n(x)]\omega(t)/\omega(x)$  and its derivatives of various orders, where  $\omega(t) = \prod_{i=0}^n (t - x_i)$ .)

4. a) Determine the coefficients  $a, b, c, d$  in the following quadrature formula

$$\int_{-1}^1 f(x) dx \approx a f(-1) + b f(1) + c f(-1) + d f(1)$$

such that this formula is exact for polynomials of the highest possible degree.

b) Derive an error bound using any relevant interpolation error formula that you know of.

5. a) State a convergence criterion for the fixed point iteration  $x_{n+1} = F(x_n)$ .

b) Let  $p > 0$  be a given number and consider the iterations  $x_{n+1} = \sqrt{p + x_n}$  for  $n = 0, 1, 2, \dots$  with an initial point  $x_0$ . Prove that if  $x_0 > -p + \frac{1}{4}$ , then the iterations converge.

c) Write a pseudo-code implementing fixed point iterations with the following stopping criteria:  $|x_{n+1} - x_n| < \epsilon_1 |x_n|$  OR  $|f(x_{n+1})| < \epsilon_2$  OR  $n + 1 > N_{\max}$  where  $N_{\max}$  is a specified maximum number of iterations.

6. For the initial value problem

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0,$$

determine the coefficients  $a_1, a_2, p_1, p_2$  in the following scheme

$$Y_0 = y_0,$$

$$Y_{i+1} = Y_i + h \left( a_1 f(t_i, Y_i) + a_2 f(t_i + p_1 h, Y_i + p_2 h f(t_i, Y_i)) \right) \quad i = 0, 1, 2, \dots$$

to obtain a second order method, where  $h$  is a uniform step length.