Numerical Analysis Qualifying Exam
August 21, 2008, 9:00am–1:00pm

Instructions

- Write your ISU ID Number on every page that you turn in. Do not write your name on any sheet that you turn in
- Work all 6 problems. Start each problem on a separate sheet of paper and clearly indicate the problem number

1. Let A be an $n \times n$ matrix and $\mathbf{b}$ an n-dimensional vector.

   a) Write out the Jacobi iteration formula for solving $A\mathbf{x} = \mathbf{b}$.

   b) Suppose that A has the row diagonal-dominant property:

$$a_{ii} > \sum_{j=1, j \neq i}^{n} |a_{ij}|, \qquad i = 1, \ldots, n.$$

   Prove the convergence of the Jacobi iterations. (You are allowed to use a convergence criterion involving the norm of the iteration matrix.)

2. Let A be an $n \times n$ matrix and $\mathbf{x}$ an n-dimensional vector. Suppose that A is upper triangular with an upper bandwidth p (total bandwidth including the main diagonal is $p + 1$).

   a) Write a pseudocode (or a real code in MATLAB or another programming language) to compute $A\mathbf{x}$ in $\leq (2p + 1)n$ floating point operations with A stored as an $n \times n$ matrix. Verify the total number of floating point operations in your code is indeed no more than $(2p + 1)n$. (Note that the operations $+, -, \times, /, <, >$ are all counted as floating point operations, but the automatic loop index advancing and intrinsic min or max function evaluation should not be counted.)

   b) Rewrite the code in a) with matrix A stored in a compact form, e.g., A is stored in a $n \times (p+1)$ matrix. Illustrate your storage method using a 5×5 upper triangular matrix with 2 upper bands and write a pseudocode script to define the compact storage matrix for your illustrative example.

3. Given a $C[a,b]$ function f and a partition of $[a,b]$: $a = x_0 < x_1 < x_2 < \cdots < x_n = b$,

a) Let pwL(x) be the piecewise linear interpolant of f on $[a,b]$. Write out the (piecewise) explicit formulae for pwL(x).

b) Assume $f \in C^2[a,b]$. Prove $|f(x) - \mathrm{pwL}(x)| \le \max_{[a,b]} |f''(x)|\, h^2$ where $h = \max_{1\le i\le n} |x_i - x_{i-1}|$. Of course, you are not allowed to simply invoke Lagrange interpolation error formulae. (Hint: on the interval $[x_i, x_{i+1}]$, write pwL(x) in the form of $f(x_i) + d_i(x - x_i)$ and use Taylor expansions on $f(x)$ and $f(x_{i+1})$.)

c) Write a pseudocode (or a real code in MATLAB or another programming language) for evaluating pwL(x) and for plotting the error function $|f(x) - \mathrm{pwL}(x)|$. (You may invoke $f(x)$ as if it were an intrinsic function.)

4. Consider Gaussian quadrature

$$\int_{-1}^{1} f(x)\, dx \approx \sum_{i=1}^{n} \alpha_i\, f(x_i)$$

where $x_1, x_2, \cdots, x_n$ are the roots of the orthogonal Legendre polynomial $L_n(x)$ and the quadrature formula is exact for polynomials of degree $\le (2n-1)$.

a) Derive the following formulae for α_i by requiring the quadrature formula to be exact for polynomials of degree $\le (n-1)$:

$$\alpha_j = \int_{-1}^{1} \ell_j(x)\, dx$$

where $\ell_j(x)$, $j = 1, 2, \cdots, n$, are the $(n-1)$-degree Lagrange interpolation basis polynomials on the interpolation nodes $x_1, x_2, \cdots, x_n$.

b) Derive the three-point Gaussian quadrature formula. (Hint: $L_n(x) = \frac{d^n[(x^2-1)^n]}{dx^n}$.)

c) Verify that the three-point Gaussian quadrature formula is exact for polynomials of degree ≤ 5.

5. Let f be a $C^3[a,b]$ (or $C^2[a,b]$) function with a single zero $z \in (a,b)$.

Given $x_0 \in (a, b)$, define the Newton's sequence

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad n = 0, 1, 2, \cdots$$

Prove that Newton's sequence converges to z provided x_0 is chosen to be sufficiently close to z. (Hint: write $z = g(z)$ where $g(x) = x - f(x)/f'(x)$ and $x_{n+1} = g(x_n)$.)

6. For the initial value problem

$$\frac{dy}{dt} = f(t, y), \quad t_0 \leq t \leq T, \quad y(t_0) = y_0,$$

consider the following Runge-Kutta method:

$$Y_0 = y_0,$$
$$Y_{i+1} = Y_i + hf(t_i + \tfrac{h}{2}, Y_i + \tfrac{h}{2} f(t_i, Y_i)), \quad i = 0, 1, 2, \cdots$$

Using Taylor expansion techniques to prove that this method is of second order.