NUMERICAL ANALYSIS QUALIFYING EXAM

Fall 2007
Saturday, August 18 2007, 9:00 am - 12:00 noon

Instructions

- Write your complete social security number on every page that you turn in. Do NOT write your name on any sheet that you turn in.
- Work all 6 problems. Start each problem on a separate sheet of paper, and clearly indicate the problem number.

1. Let A and B be $n \times n$ matrices and assume that B is symmetric and positive definite. Let $\|\cdot\|_B$ denote the norm $\|v\|_B = (Bv, v)^{1/2}$, for all $v \in R^n$, where $(\cdot, \cdot)$ denotes the dot product on $R^n \times R^n$. Assume that there are positive constants c_0 and c_1 satisfying

$$c_0 \|v\|_B^2 \le (Av, v), \quad \forall v \in R^n$$

and

$$(Av, w) \le c_1 \|v\|_B \|w\|_B \quad \forall v, w \in R^n.$$

(a) Show that $\|Av\|_{B^{-1}} \le c_1 \|v\|_B \quad \forall v \in R^n$,
(b) Consider the iterative method

$$x_{n+1} = x_n + \tau B^{-1}(f - Ax_n)$$

for solving $Ax = f$ and set $e_n = x - x_n$. Use Part (a) above to show that for $0 < \tau \le c_0/c_1^2$,

$$\|e_{n+1}\|_B \le (1 - c_0\tau)\|e_n\|_B.$$

2. Assume that the matrix A has an eigenvalue λ with eigenvector v, and the other eigenvalues of A are some distance away from λ. Assume μ is a number close to, but not equal to λ.

Starting from an initial guess x_0, consider the shifted inverse power method

$$(A - \mu I)y_{k+1} = x_k$$
$$x_{k+1} = y_k / \|y_k\|.$$

and answer the following questions:
(a) What vector does x_k converge to?
(b) What is the order of convergence?
(c) How do you get an estimate of λ?
(d) Why is the second step (normalization) of the algorithm necessary?
(e) The matrix $(A - \mu I)$ is almost singular, and therefore has a very large condition number. How does that influence the accuracy of this method?

3. Find the natural cubic spline through the points $(0,0)$, $(2,1)$, $(3,0)$.

Note: Nobody expects you to have the spline formulas memorized. You have enough time to set up the spline as a piecewise polynomial with arbitrary coefficients, derive the necessary equations from scratch, and solve them by hand.

4. Let $f \in C_1[a,b]$, and suppose that $x_1, \cdots x_n$ are pairwise distinct points. Show that for every $\epsilon > 0$, there exists a polynomial p such that $\|f - p\|_\infty < \epsilon$ and, simultaneously, satisfies the interpolation condition $p(x_k) = f(x_k), \quad 1 \leq k \leq n$.

5. (a) Construct a one-parameter family of implicit linear 2-step methods

$$\sum_{j=0}^{2} \alpha_j y_{n+j} = h \sum_{j=0}^{2} \beta_j f_{n+j},$$

of greatest possible order, and find the error constant. Use the coefficient α_1 as the parameter and determine $\alpha_0, \beta_0, \beta_1, \beta_2$ in terms of α_1. Assume the standard normalization $\alpha_2 = 1$.

(b)For what range of the parameter α_1 are the methods you found in the previous problem convergent? Can you increase the order and still have a convergent method by an appropriation choice of α_1?

6. Let $\alpha \in \mathbb{R}$ be a fixed point of the real valued function $\phi(x)$ and assume that $\phi(x)$ satisfies:

- $\phi \in C^p(I_\delta)$, where $p \geq 2$ and $I_\delta = [\alpha - \delta, \alpha + \delta]$ for some $\delta > 0$.
- $\max\{|\phi'(x)| : x \in I_\delta\} < 1$.
- $\phi^{(k)}(\alpha) = 0$ for $k = 1, \cdots, p-1$, and $\phi^{(p)}(\alpha) \neq 0$.

Show that if $x_0 \in I_\delta$ then the fixed point iteration $x_{n+1} = \phi(x_n)$ converges and determine the order and asymptotic error constant of convergence. Apply your results to Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$