(1) Let $u$ and $v$ be column vectors, and consider the rank one perturbation of the identity defined by $A = I + uv^*$. 

(a) Show that if $A$ is nonsingular, then its inverse has the form $A^{-1} = I + \alpha uv^*$ for some scalar $\alpha$. Give an expression for $\alpha$.

(b) For what $u$ and $v$ is $A$ singular? Show that if $A$ is singular then it is a projector. For what $u$ and $v$ is $A$ an orthogonal projector?

(2) Let $T$ be an $(n \times n)$ matrix and $\{x^{(k)}\}$ a sequence of column vectors defined inductively by $x^{(k)} = T x^{(k-1)}$, for a given $x^{(0)}$.

(a) Given a vector norm $\| \cdot \|$, let $\| \cdot \|$ also denote the matrix norm induced by this vector norm. What condition on $\|T\|$ will guarantee that the sequence $\{x^{(k)}\}$ will converge, for any choice of $x^{(0)}$?

(b) Consider the concrete example

$$
T = \begin{bmatrix}
0.2 & -0.3 & 0.35 & 0.5 \\
0.1 & 0.2 & 0.25 & 0 \\
0 & -0.1 & -0.1 & 0.2 \\
-0.5 & 0.1 & 0.1 & 0.1
\end{bmatrix}
$$

(so $n = 4$). Compute $\|T\|_\infty$. Show that for any starting vector $x^{(0)}$, the sequence $\{x^{(k)}\}$ is convergent in the norm $\| \cdot \|_\infty$. Do these facts contradict your answer to the first part of this problem?
(3) Let \(a = x_0 < x_1 < x_2 < \cdots < x_n = b\) be a partition of the interval \([a, b]\), let \(f\) be a twice continuously-differentiable function on \([a, b]\), and let \(g\) be the piecewise linear interpolant of \(f\) (i.e., \(g\) is linear on each subinterval \([x_i, x_{i+1}]\), \(i = 0, 1, \ldots, n-1\), and \(g(x_i) = f(x_i), \ i = 0, 1, \ldots, n\)).

(a) Derive a numerical quadrature formula for \(\int_a^b f(x) \, dx\) by approximating this integral with \(\int_a^b g(x) \, dx\).

(b) Show that the error in the approximation satisfies
\[
\left| \int_a^b f(x) \, dx - \int_a^b g(x) \, dx \right| \leq Ch^2,
\]
where \(h\) is the maximum step length and \(C\) is a constant that is independent of \(h\).

(4) Assume that \(f : \mathbb{R} \to \mathbb{R}\) is a smooth function which satisfies
\[
f'(x) > 0, \quad f''(x) > 0 \quad \text{for all } x \in \mathbb{R},
\]
and has a root \(x^*\) (i.e. \(f(x^*) = 0\)). Prove that Newton’s method converges to \(x^*\) for any initial guess \(x_0 \in \mathbb{R}\).

(5) Let \(f(x)\) be a smooth function on an interval \((a, b)\), \(x_0 < x_1 < x_2\) be three points in \((a, b)\), and \(p_2(x)\) be the interpolating polynomial of degree 2 for \(f(x)\) at \(\{x_0, x_1, x_2\}\). Suppose that \(f''(x_1)\) is to be approximated by \(p_2''(x_1)\). Derive a formula for this in terms of the quantities \(f_0, f_1, f_2, h_1, h_2\), where \(f_k = f(x_k)\) and \(h_k = x_k - x_{k-1}\). Also determine an error bound in terms of \(h = \max\{h_1, h_2\}\) and derivatives of \(f(x)\). (You may use the fact that \(\frac{d}{dx}f[x_0, x_1, x_2, x] = f[x_0, x_1, x_2, x, x]\).)

(6) The differential equation \(y' = f(x, y)\) can be approximated by the finite difference scheme
\[
y_{n+1} = y_n + \frac{h}{2}[y_n + y'_{n+1}] + \frac{h^2}{12}[y''_n - y''_{n+1}],
\]
where \(y'_{n+1} = f(x_n, y_n)\) and \(y''_n = f_x(x_n, y_n) + f(x_n, y_n)f_y(x_n, y_n)\).

(a) Show that this scheme is fourth-order accurate.

(b) State what it means for \(h\lambda\) to belong to the region of absolute stability for this scheme, and show that the region of absolute stability contains the entire negative real axis.