NUMERICAL ANALYSIS QUALIFYING EXAM

Fall 2006
Saturday, August 19 2006, 9:00 am - 12:00 noon
Room 305 Carver

Instructions:

- Write your complete student identification number on every page that you turn in. **DO NOT** write your name on any sheet that you turn in.
- Work all 6 problems. Start each problem on a separate sheet of paper, and clearly indicate the problem number.

(1) Let u and v be column vectors, and consider the *rank one perturbation of the identity* defined by $A = I + uv^*$.

(a) Show that if A is nonsingular, then its inverse has the form $A^{-1} = I + \alpha uv^*$ for some scalar α. Give an expression for α.

(b) For what u and v is A singular? Show that if A is singular then it is a projector. For what u and v is A an orthogonal projector?

(2) Let T be an $(n \times n)$ matrix and $\{x^{(k)}\}$ a sequence of column vectors defined inductively by $x^{(k)} = Tx^{(k-1)}$, for a given $x^{(0)}$.

(a) Given a vector norm $\|\cdot\|$, let $\|\cdot\|$ also denote the matrix norm induced by this vector norm. What condition on $\|T\|$ will guarantee that the sequence $\{x^{(k)}\}$ will converge, for any choice of $x^{(0)}$?

(b) Consider the concrete example

$$T = \begin{bmatrix} 0.2 & -0.3 & 0.35 & 0.5 \\ 0.1 & 0.2 & 0.25 & 0 \\ 0 & -0.1 & -0.1 & 0.2 \\ -0.5 & 0.1 & 0.1 & 0.1 \end{bmatrix}$$

(so $n = 4$). Compute $\|T\|_\infty$. Show that for any starting vector $x^{(0)}$, the sequence $\{x^{(k)}\}$ is convergent in the norm $\|\cdot\|_\infty$. Do these facts contradict your answer to the first part of this problem?

(3) Let $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ be a partition of the interval $[a, b]$, let f be a twice continuously-differentiable function on $[a, b]$, and let g be the piecewise linear interpolant of f (i.e., g is linear on each subinterval $[x_i, x_{i+1}]$, $i = 0, 1, \cdots, n-1$, and $g(x_i) = f(x_i)$, $i = 0, 1, \cdots, n$).

(a) Derive a numerical quadrature formula for $\int_a^b f(x)\,dx$ by approximating this integral with $\int_a^b g(x)\,dx$.

(b) Show that the error in the approximation satisfies

$$\Big|\int_a^b f(x)\,dx - \int_a^b g(x)\,dx\Big| \le Ch^2,$$

where h is the maximum step length and C is a constant that is independent of h.

(4) Assume that $f : \mathbb{R} \to \mathbb{R}$ is a smooth function which satisfies

$$f'(x) > 0, \quad f''(x) > 0 \qquad \text{for all } x \in \mathbb{R},$$

and has a root x^* (i.e. $f(x^*) = 0$). Prove that Newton's method converges to x^* for any initial guess $x_0 \in \mathbb{R}$.

(5) Let $f(x)$ be a smooth function on an interval (a, b), $x_0 < x_1 < x_2$ be three points in (a, b), and $p_2(x)$ be the interpolating polynomial of degree 2 for $f(x)$ at $\{x_0, x_1, x_2\}$. Suppose that $f''(x_1)$ is to be approximated by $p_2''(x_1)$. Derive a formula for this in terms of the quantities f_0, f_1, f_2, h_1, h_2, where $f_k = f(x_k)$ and $h_k = x_k - x_{k-1}$. Also determine an error bound in terms of $h = \max\{h_1, h_2\}$ and derivatives of $f(x)$. (You may use the fact that $\frac{d}{dx} f[x_0, x_1, x_2, x] = f[x_0, x_1, x_2, x, x]$.)

(6) The differential equation $y' = f(x, y)$ can be approximated by the finite difference scheme

$$y_{n+1} = y_n + \frac{h}{2}[y_n' + y_{n+1}'] + \frac{h^2}{12}[y_n'' - y_{n+1}''],$$

where $y_n' = f(x_n, y_n)$ and $y_n'' = f_x(x_n, y_n) + f(x_n, y_n) f_y(x_n, y_n)$.

(a) Show that this scheme is fourth-order accurate.

(b) State what it means for $h\lambda$ to belong to the region of absolute stability for this scheme, and show that the region of absolute stability contains the entire negative real axis.