Numerical Analysis Qualifying Exam, Fall 2003 **Last 4 Digits of #SS** ____________

1. (10 points) Let $A = LU$ be the LU factorization of an $n \times n$ matrix A with $|l_{i,j}| \le 1$. Let a_i and u_i denote the ith row of A and U, respectively. Verify the equation

$$u_i = a_i - \sum_{j=1}^{i-1} l_{i,j} u_j$$

and show that $\| U \|_\infty \le 2^{n-1} \| A \|_\infty$, where $\| A \|_\infty = \max_i \sum_j | a_{i,j} |$.

2. (10 points) Let $x_0 \in R^n$ be a starting point and $p_0, p_1, \ldots, p_{n-1} \in R^n$ be a set of conjugate directions of a symmetric positive definite matrix A. Show that the following iteration

$$x_{k+1} = x_k + a_k p_k, \qquad a_k = (b - Ax_k)^T p_k / (p_k^T A p_k)$$

solves the equation $Ax = b$ in at most n steps.

3. (10 points) Let $f : R \to R$ be a function with a simple zero x_*. Assume that f is twice continuously differentiable for all x in a neighborhood of x_*, $U(x_*) = \{x : | x - x_* | \le r\}$. Show that the following iteration

$$y = x_k - f'(x_k)^{-1} f(x_k), \qquad x_{k+1} = y - f'(x_k)^{-1} f(y)$$

converges to x_* locally at least cubically.

4. (10 points) The Bernstein polynomial of degree n for $f \in C[0,1]$ is given by

$$B_n(x) = \sum \binom{n}{k} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k},$$

where $\binom{n}{k} = n!/k!(n-k)!$. These polynomials can be used to approximate f (Weierstrass approximation) since $\lim_{n\to\infty} B_n(x) = f(x)$, for $x \in [0,1]$.

(a). Show for any integer $n > 0$, $B_n(x)$ provides faithful approximation to $f(x) = 1$ and $f(x) = x$, i.e., $B_n(x) = f(x)$.

(b). Use part (a) to show that, for $f(x) = x^2$, $B_n(x) = x^2 + x(1-x)/n$.

(c). Use part (b) to estimate the value of n necessary for $| B_n(x) - x^2 | \le 10^{-6}$ to hold for all $x \in [0,1]$.

5. (10 points) The basic trapezoid rule to compute the finite integral $I(f) = \int_a^b f(x)dx$ is

$$T_1(f) = \frac{1}{2}(b-a)[f(b) + f(a)].$$

(a). Show for any polynomial of degree 1, $f = p_1(x) = Ax + B$, $I(f) = T_1(f)$.

(b). Use (a) and Taylor's theorem to show the identity

$$I(f)-T_1(f)=\frac{1}{2}\int_a^b (a-t)(b-t)f''(t)dt,$$

and the error estimate

$$|I(f)-T_1(f)|\le \frac{1}{8}(b-a)^2\int_a^b |f''(t)|dt.$$

[Hint: Use Taylor's expression $f(x)=f(a)+f'(a)(x-a)+\int_a^x (x-t)f''(t)dt$.]

(c). Give the formula for the composite trapezoid rule $T_n(f)$ with n-subintervals, and decide whether the above error estimate is still valid.

6. (10 points) Consider the general linear multi-step method

$$\sum_{j=0}^{k} a_j y_{n+j} = h\sum_{j=0}^{k} b_j f_{n+j},$$

having *k*-steps.

(a). Show that if $y(x)$ is analytic then the linear difference operator associated with the method

$$L[y(x):h]=\sum_{j=0}^{k} a_j y(x+jh)-h\sum_{j=0}^{k} b_j y'(x+jh)$$

has the expansion $L[y(x):h]=\sum_{m=0}^{\infty} C_m y^{(m)}(x)h^m$, where

$$C_0=\sum_{j=0}^{k} a_j \text{ and } C_m=\sum_{j=0}^{m}\left(\frac{j^m a_j}{m!}-\frac{j^{m-1}b_j}{(m-1)!}\right), m\ge 1.$$

(b). Construct a one-parameter family of implicit linear 2-step methods, of greatest possible order. In addition to the order, determine the error constant. Use the coefficient a_1 as the parameter to determine a_0, b_0, b_1, b_2 and the error constant in terms of a_1. Use the normalization $a_2=1$.