ANALYSIS QUALIFYING EXAMINATION

Spring, 2008

Thursday, January 10, 2008

305 Carver Hall, 9:00am-1:00pm

Instructions:

- Write your university identification number on every page that you turn in. Do NOT write your name on any page that you turn in.
- Your score will be based on 6 total problems. Work no more than 6 problems. No credit will be awarded for more than 6 problems. If you turn in more than 6 problems, then all will be graded and your score will be the total of the scores on the 6 lowest scoring problems.
- Work each problem on a separate piece of paper and clearly indicate the problem and problem part on each page.
- To pass you must work on at least 2 problems from Part I and at least 3 problems from Part II, and receive substantial credits from both parts. In the grading, one completely correct solution will be counted as more than two "half correct" solutions.
- Every effort is made to proofread the exam, but misprints may occur. If you believe that a problem has been stated incorrectly, check with the proctor and indicate your interpretation in the solution. Do not interpret the problem in a way that it becomes trivial.

Part I. Complex Analysis

1. Let $\alpha \in \mathbb{C}$ and let
$$f(z) = \frac{\sin(2z) - \sin(2\alpha)}{(z-\alpha)^2}.$$
Write the Laurent series for f about $z = \alpha$. Give a complete characterization of the coefficient of $(z-\alpha)^k$ for all integers k.

2. Compute the number of zeroes of $f(z) = 4z^9 + 3z^6 + 2z + 1$ in the disk $\{z \in \mathbb{C} : |z| < 2\}$.

3. Draw and describe a closed path γ in the complex plane such that
$$\frac{1}{2\pi i}\int_\gamma \frac{z+12}{(z-3)(z+2)}\, dz = 13.$$

4. Prove that there is no one-to-one analytic function that maps the punctured disk $\{0 < |z| < 1\}$ onto the annulus $\{1 < |z| < 2\}$.

Part II. Real Analysis

1. Let $Q[0,1]$ be the set of rational numbers in $[0,1]$. Since $Q[0,1]$ is a countable set, there exists a 1-1, onto function $f : \mathbb{N} \to Q[0,1]$; in fact there are many such functions.
   a) Construct a 1-1, onto function $g : \mathbb{N} \to Q[0,1]$ such that $\sum_{k=1}^{\infty} g(k)^k$ diverges.
   b) Is there a 1-1, onto function $h : \mathbb{N} \to Q[0,1]$ such that $\sum_{k=1}^{\infty} h(k)^k$ converges?

2. Suppose $E \in \mathbb{R}$ is a Borel set with finite measure and suppose $f : E \to (0,\infty)$ with $\int_E f(x)dx < \infty$. Identify, with proof, $\lim_{n\to\infty} \int_E (f(x))^{1/n}\, dx$.

3. Suppose $f, g \in L^2(\mathbb{R})$. For a fixed $z \in \mathbb{R}^+$, define a funtion $h_z : \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ by
$$h_z(y) = \int_0^\infty f(x)g(y - zx)\, dx.$$
Show that h_z is continuous.

4. Let $\lambda \times \lambda$ denote the Lebesgue measure on $\mathbb{R}^2$. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is bounded and measurable, and for each open disk D in $\mathbb{R}^2$,
$$\int_D f\, d(\lambda \times \lambda) = 0.$$
Show that $f = 0$ $\lambda \times \lambda$ a.e.

5. Let (X, Σ, μ) be a measure space and let $f : X \to [0, +\infty]$ be a μ-measurable function. Given a set $D \in \Sigma$ with $\mu(D) < \infty$, define
$$D_n = \{x \in D : f(x) \geq n\}.$$
Prove that $\int_D f\, d\mu < \infty$ if and only if $\sum_{n=1}^{\infty} \mu(D_n) < \infty$.

6. Suppose μ is a measure defined on the Lebesgue measurable subsets of $[0,1]$, and let λ denote Lebesgue measure. Show that there exists a constant M such that for every $f \in L^1[0,1]$,
$$\int_0^1 |f| d\mu \leq M \int_0^1 |f| d\lambda$$
if and only if $\mu << \lambda$ and $\dfrac{d\mu}{d\lambda} \in L^\infty[0,1]$.