

ANALYSIS QUALIFYING EXAMINATION

Spring, 2007

Saturday, January 13, 2007, 9:00 am - 12:00 noon

Room 305 Carver

Instructions:

- Write your university identification number on every page that you turn in. Do **NOT** write your name on any page that you turn in.
- Work no more than 6 problems. No credit will be awarded for more than 6 problems. If you turn in more than 6 problems, then all will be graded and your score will be the total of the scores on the 6 lowest scoring problems.
- Work each problem on a separate piece of paper and clearly indicate the problem and problem part on each page.
- To pass you must work on at least 2 problems from Part I and at least 3 problems from Part II, and receive substantial credits from both parts. In the grading, one completely correct solution will be counted as more than two "half correct" solutions.

Part I. Complex Analysis

1. Suppose f is analytic and non-vanishing on an open set U . Prove that $\log |f|$ is harmonic on U .

2. Evaluate the integral

$$\int_0^{\pi/2} \frac{dx}{9 + 7 \sin^2 x},$$

(Hint: $\sqrt{2304} = 48$.)

3. Let f be analytic on an open set U , $z_0 \in U$ and $f'(z_0) \neq 0$. Show that there exists $r > 0$ such that $\{z : |z - z_0| \leq r\} \subset U$ and

$$\frac{2\pi i}{f'(z_0)} = \int_C \frac{1}{f(z) - f(z_0)} dz,$$

where $C = \{z : |z - z_0| = r\}$.

4. How many roots does the equation $z^7 - 2z^5 + 6z^3 - z + 1 = 0$ have in the disk $|z| < 1$?

Part II. Real Analysis

1. Let (X, \mathcal{B}, μ) be a (positive) measure space with $\mu(X) < \infty$. Suppose f and g are real-valued measurable functions on X such that $\int_X f d\mu = \int_X g d\mu$. Show that either (a) $f = g$ a.e. or (b) there exists $E \in \mathcal{B}$ such that $\int_E f d\mu > \int_E g d\mu$.

2. Suppose $A \subset \mathbf{R}$ is Lebesgue measurable and satisfies $m(A \cap (a, b)) \leq \frac{b-a}{2}$ for all $a < b$, where m is the Lebesgue measure on \mathbf{R} . Prove that $m(A) = 0$.

3. We say that a real-valued function f defined on an interval I is *Lipschitz* if there is a constant M such that

$$|f(x) - f(y)| \leq M|x - y|$$

for all x and y in I .

- (a) Show that a Lipschitz function is absolutely continuous.
(b) Give an example of an absolutely continuous function which is not Lipschitz.

4. Suppose $1 \leq p < p' < \infty$. Show that $L^{p'}[0, 1] \subset L^p[0, 1]$ and $L^{p'}[0, 1] \neq L^p[0, 1]$.

5. Suppose $f(x, y)$ is a bounded measurable function on $[0, 1] \times [0, 1]$. Show that if for every $a < b$ and $c < d$

$$\int_{[a,b]} \int_{[c,d]} f(x, y) dx dy = 0$$

then $f = 0$ a.e.

6. Let X be the metric space (\mathbf{R}, d) , where \mathbf{R} is the set of real numbers and

$$d(x, y) = \frac{|x - y|}{1 + |x - y|}.$$

Show that there is a decreasing sequence of nonempty closed bounded sets with empty intersection.