**ANALYSIS QUALIFYING EXAMINATION**

Spring, 2004
Saturday, January 10, 2004 9:00-12:00
Room 408 Carver

Instructions:

- **Write your social security number on every page you hand in.** Do **NOT** write your name on any sheet you turn in.
- Work no more than 6 problems. No credit will be given for more than 6 problems so if you turn in more than 6 problems, all problems will be graded and your total score will be based on your 6 worst problem scores.
- Work each problem on a separate piece of paper and clearly indicate the part and the problem number of that part.
- To pass you must work at least two problems from Part I and three problems from Part II. One correct problem will be counted more than two "half correct" problems in the grading.

1. The function $f(\cdot)$ has a double pole at $z = 0$ with residue 2, a simple pole at $z = 1$ with residue 2 and is otherwise analytic. Suppose that $f(\cdot)$ is bounded at $z = \infty$ and $f(2) = 5$ while $f(-1) = 2$. What is $f(z)$?

2. Does there exist a one to one analytic mapping of the unit disk onto the complex plane? Either give an example or show that no such mapping exists.

3. Evaluate

$$\oint_{|z|=1} e^{1/\sin z}\, dz$$

where the orientation is in the counterclockwise direction.

4. How many roots does $f(z) = 2z^5 - 6z^2 + z + 1$ have in the annulus $\{z | \ 1 \le |z| \le 2\}$?

**PART II. Real Analysis**

1. Suppose $c > 0$ is given. Consider the set of all real valued, measurable functions, $f(\cdot)$, on a fixed measure space $(X, \mathfrak{M}, \mu)$ such that

$$\int_X f^2\, d\mu = \int_X f^3\, d\mu = \int_X f^4\, d\mu = c.$$

Characterize the elements of this set. Hint: Evaluate

$$\int_X [f(s)(1 - f(s))]^2\, d\mu(s).$$

2. Find, for $\alpha \ge 1$,

$$\lim_{n\to+\infty} \int_0^\pi n \ln\left[1 + \left(\frac{\sin x}{n}\right)^\alpha\right] dx.$$

Be sure you justify any changes of limit and integration.

3. Let $m =$ Lebesgue measure.

   A. Assume that $E_1 \cup E_2$ is Lebesgue measurable. Show that if $E_2$ has measure 0, then $E_1$ is Lebesgue measurable.

   B. Let $X = [0, 1]$, $\mu =$ counting measure.

      a. Show that $m \ll \mu$, but $dm \neq f d\mu$ for any $f$.

      b. Why does this example not contradict the Radon-Nikodym theorem?

4. Let $\{q_n\}$ be an enumeration of the rationals in $[0, 1]$. Define the function $f$ on $[0, 1]$ by

$$f(x) = \sum_{n \text{ with } q_n < x} 2^{-n}.$$

   a. Where is this function continuous/discontinuous?
   b Show that this function is Riemann integrable.
   c. What is the value of the integral?

5. Suppose that $f : [0, 1] \to \mathbb{R}$ is continuous, $f(0) = 0$ and $f'(0)$ exists.

   a. Show that $x^{-\frac{3}{2}} f(x)$ is in $\mathfrak{L}^p$ if $1 \leq p < 2$.
   b. Show that the first statement fails if the hypothesis that $f'(0)$ exists is omitted, the other assumptions being unchanged.

6. Let $m$ be Lebesgue measure on $[0, 1]$. A sequence $\{f_n\}_{n=0}^{\infty}$ of measureable functions is said to converge to $f$ *in measure* if for every $\epsilon > 0$, there is an index $N$ such that $m(\{x| \, |f_n(x) - f(x)| > \epsilon\}) < \epsilon$ if $n > N$. Prove the following:

   a. If the sequence converges pointwise to $f$ a. e. then it converges in measure to $f$.
   b. If the sequence converges in $L^1(0, 1)$, then it converges in measure to $f$.
   c. Give counter examples to show that the converses of parts a, b fail.

7. If $\lambda, \mu$ are two Borel measures on $R^1$, we define their convolution as follows:

$$\lambda * \mu(E) = \lambda \times \mu(\{(x, y)|x + y \in E\})$$

for every Borel measurable set $E$ in on the line. Given that if $E$ is a Borel measurable set, then $\{(x, y)|x + y \in E\}$ is Borel measurable. Show that

$$\lambda * \mu(E) = \int_{R^1} \lambda(E - t) \, d\mu(t).$$

Suppose that $\lambda, \mu$ are absolutely continuous with respect to Lebesgue measure. Is the same true of $\lambda * \mu$? If so, give a proof. If not, give a counter example.