ANALYSIS QUALIFYING EXAM

Spring 2003
January 18, 2003, 9:00 am - 12:00 noon
Room 408 Carver

Instructions

- Write your complete social security number on every page that you turn in. Do NOT write your name on any sheet that you turn in.
- Work 6 problems, with at least 2 from Part I and at least 3 from Part II. No credit will be given for additional problems, and if additional problems are turned in, only the worst ones will be counted.
- Work each problem on a separate sheet of paper, and clearly indicate the part and problem number.
- To pass, you must receive substantial credit from each part. One correct problem will be counted more than two "half correct" problems in the grading.

Part I: Complex Analysis

1. Evaluate

$$\int_C \frac{y\,dx - x\,dy}{x^2+y^2}$$

around the contour (in the counterclockwise direction) whose point set is $\{(x,y) : |x|+|y| = \pi\}$.

2. Find a conformal mapping from $S = \{z : |z-1| > 1 \text{ and } |z-2| < 2\}$ onto the unit ball $\{z : |z| < 1\}$.

3. Let $a \in \mathbb{R}$, $a > 1$. Prove that the equation

$$ze^{a-z} = 1$$

has exactly one solution in the unit ball $\{z : |z| < 1\}$, and this solution is real and positive.

4. Let $\mathcal{D}$ be an open set containing the closed unit disk $\{z : |z| \leq 1\}$ in $\mathbb{C}$. If f is analytic in $\mathcal{D}$, and $\Im f = 0$ for $|z| = 1$, show that f is constant and real.

Part II: Real Analysis

1. Show that

$$f(x) = \begin{cases} x \sin \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is continuous on $[-1, 1]$ but is not of bounded variation there. Is

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

of bounded variation? Why or why not?

2. Show that if

$$\lim_{h \to 0} \frac{1}{h} \int_a^b |f(x+h) - f(x)| dx = 0,$$

then $f(x)$ =constant almost everywhere.

3. Prove the following statement:
Given a Lebesgue measurable extended real-valued function f on $[0, 1]$ which takes the values $\pm\infty$ only on a set of measure 0, and given $\epsilon > 0$, there is an M such that $|f| \leq M$ except on a set of measure less than ϵ.

4. Let $A \subset \mathbb{R}$ have Lebesgue measure zero and let

$$B = \{x \in \mathbb{R} : x = y^2 \text{ for some } y \in A\}.$$

Show that B also has Lebesgue measure zero.

5. A Banach space is called *separable* if it contains a countable, dense set. Show that ℓ_1 is separable, ℓ_∞ is not.

6. Let $(X, \mathcal{M})$ be a σ-finite measure space, and λ, μ positive measures on $(X, \mathcal{M})$. Show that the following statements are equivalent.

(1) For every $E \in \mathcal{M}$, $\mu(E) = 0$ implies $\lambda(E) = 0$.

(2) For every $\epsilon > 0$ there exists a $\delta > 0$ such that $\lambda(E) < \epsilon$ whenever $E \in \mathcal{M}$, $\mu(E) < \delta$.